3 ACCELERATION ANALYSIS

Introduction

Velocity of a moving body is a vector quantity having magnitude and direction. A change in the velocity requires any of the following conditions to be fulfilled:

- A change in the magnitude only
- A change in the direction only
- A change in both magnitude and direction

The rate of change of velocity with respect to time is known as *acceleration* and it acts in the direction of the change in velocity. Thus acceleration is also a vector quantity. To find linear acceleration of a point on a link, its linear velocity is required to be found first. Similarly, to find the angular acceleration of a link, its angular velocity has to be found. Apart from the graphical method, algebraic methods are also discussed in this chapter. After finding the accelerations, it is easy to find inertia forces acting on various parts of a mechanism or machine.

3.1 ACCELERATION

Let a link OA, of length r, rotate in a circular path in the clockwise direction as shown in Fig. 3.1(a). It has an instantaneous angular velocity ω and an angular acceleration α in the same direction, i.e., the angular velocity increases in the clockwise direction.

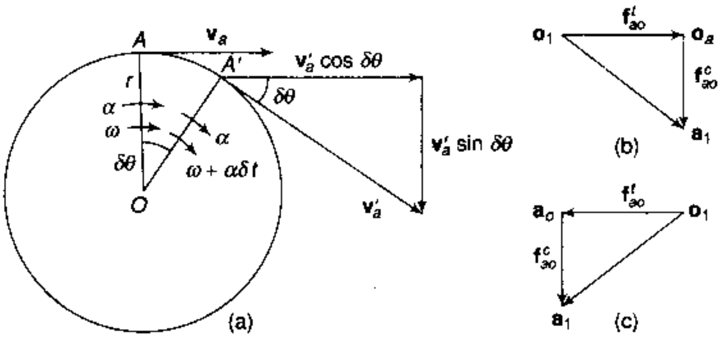


Fig. 3.1

Tangential velocity of A, $v_a = \omega r$

In a short interval of time δt, let OA assume the new position OA' by rotating through a small angle $\delta\theta$.

Angular velocity of OA', $\omega'_a = \omega + \alpha\,\delta t$

Tangential velocity of A', $v'_a = (\omega + \alpha\,\delta t)\,r$

The tangential velocity of A' may be considered to have two components; one perpendicular to OA and the other parallel to OA.

Change of Velocity Perpendicular to OA

Velocity of $A \perp$ to $OA = v_a$

Velocity of $A' \perp$ to $OA = v'_a \cos\delta\theta$

$\therefore$ change of velocity $= v'_a \cos\delta\theta - v_a$

$$\text{Acceleration of } A \perp \text{ to } OA = \frac{(\omega + \alpha\,.\delta t)r\cos\delta\theta - \omega r}{\delta t}$$

In the limit, as $\delta t \to 0$, $\cos\delta\theta \to 1$

$\therefore$ acceleration of $A \perp$ to $OA = \alpha\, r$

$$= \left(\frac{d\omega}{dt}\right) r \qquad \ldots\left(\alpha = \frac{d\omega}{dt}\right)$$

$$= \frac{dv}{dt} \tag{3.1}$$

This represents the rate of change of velocity in the tangential direction of the motion of A relative to O, and thus is known as the *tangential acceleration* of A relative to O. It is denoted by f^t_{ao}.

Change of Velocity Parallel to OA

Velocity of A parallel to $OA = 0$

Velocity of A' parallel to $OA = v'_a \sin\delta\theta$

$\therefore$ change of velocity $= v'_a \sin\delta\theta - 0$

$$\text{Acceleration of } A \text{ parallel to } OA = \frac{(\omega + \alpha\,\delta t)r\sin\delta\theta}{\delta t}$$

In the limit, as $\delta t \to 0$, $\sin\delta\theta \to \delta\theta$

$$\text{Acceleration of } A \text{ parallel to } OA = \omega r\frac{d\theta}{dt}$$

$$= \omega r.\omega \qquad \ldots\left(\omega = \frac{d\theta}{dt}\right)$$

$$= \omega^2 r \tag{3.2}$$

$$= \frac{v^2}{r} \ldots\ldots \quad (v = \omega r) \tag{3.3}$$

This represents the rate of change of velocity along OA, the direction being from A towards O or towards the centre of rotation. This acceleration is known as the *centripetal* or the *radial acceleration* of A relative to O and is denoted by f^c_{ao}.

Figure 3.1(b) shows the centripetal and the tangential components of the acceleration acting on A. Note the following:

1. When $\alpha = 0$, i.e., OA rotates with uniform angular velocity, $f^t_{ao} = 0$ and thus f^c_{ao} represents the total acceleration.
2. When $\omega = 0$, i.e., A has a linear motion, $f^c_{ao} = 0$ and thus the tangential acceleration is the total acceleration.
3. When α is negative or the link OA decelerates, tangential acceleration will be negative or its direction will be as shown in Fig. 3.1(c).

Total acceleration vectors are denoted by small letters with a subscript '1' attached. The meeting point of its components may be denoted by any of the small letters used for the total acceleration with a subscript of the other.

For example, components of the total acceleration $\mathbf{o_1a_1}$ can be written in either of the two ways:

1. $\mathbf{o_1o_a}$ and $\mathbf{o_aa_1}$ as in Fig. 3.1 (b)
2. $\mathbf{o_1a_o}$ and $\mathbf{a_oa_1}$ as in Fig. 3.1 (c)

Note that the centripetal acceleration is denoted by the same letters as are in the configuration diagram but the positions are changed.

3.2 FOUR-LINK MECHANISM

The configuration and the velocity diagrams of a four-link mechanism discussed in Sec. 2.5 have been reproduced in Figs 3.2(a) and (b). Let α = angular acceleration of AB at this instant, assumed positive, i.e., the speed increases in the clockwise direction.

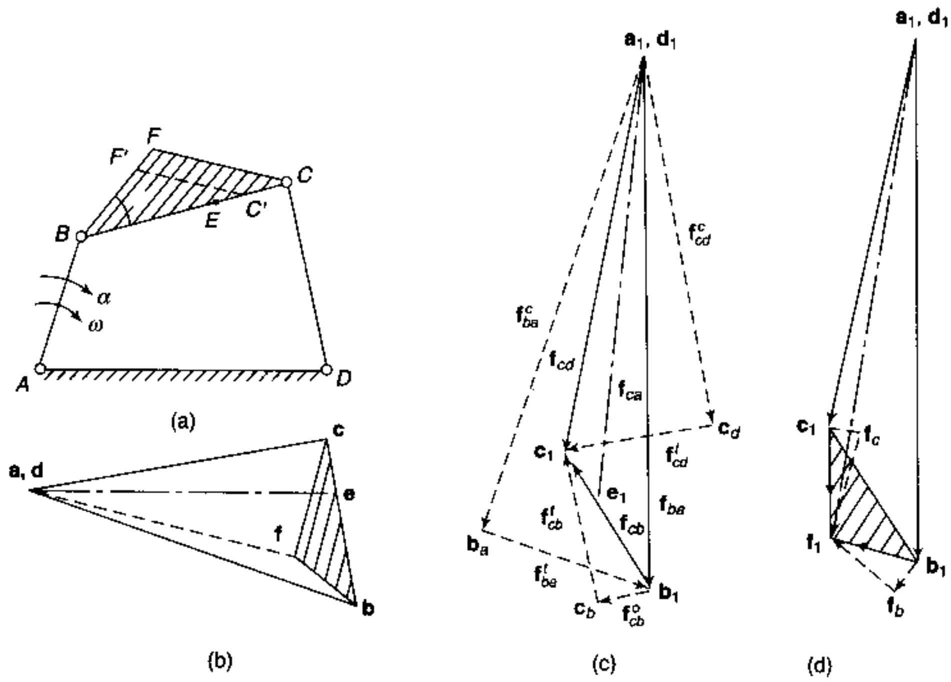


Fig. 3.2

For the construction of the acceleration diagram, a vector equation for the same can be formed similar to the one applied to the velocity diagram.

Acc. of C rel. to A = Acc. of C rel. to B + Acc. of B rel. to A

$$\mathbf{f}_{ca} = \mathbf{f}_{cb} + \mathbf{f}_{ba}$$

or

$$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

or

$$\mathbf{d}_1\mathbf{c}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{c}_1$$

Each of these accelerations may have a centripetal and a tangential component. Thus, the equation can be expanded as shown below:

$$\mathbf{f}^c_{cd} + \mathbf{f}^t_{cd} = \mathbf{f}^c_{ba} + \mathbf{f}^t_{ba} + \mathbf{f}^c_{cb} + \mathbf{f}^t_{cb}$$

or

$$\mathbf{d}_1\,\mathbf{c}_d + \mathbf{c}_d\,\mathbf{c}_1 = \mathbf{a}_1\,\mathbf{b}_a + \mathbf{b}_a\,\mathbf{b}_1 + \mathbf{b}_1\,\mathbf{c}_b + \mathbf{c}_b\,\mathbf{c}_1$$

Set the following vector table:

SN	Vector	Magnitude	Direction	Sense
1.	$\mathbf{f}^c_{ba}$ or $\mathbf{a}_1\mathbf{b}_a$	$\frac{(\mathbf{ab})^2}{AB}$	$\parallel AB$	$\rightarrow A$
2.	$\mathbf{f}^t_{ba}$ or $\mathbf{b}_a\mathbf{b}_1$	$\alpha \times AB$	$\perp AB$ or $\mathbf{a}_1\,\mathbf{b}_a$ or $\parallel$ **ab**	$\rightarrow$ **b**
3.	$\mathbf{f}^c_{cb}$ or $\mathbf{b}_1\mathbf{c}_b$	$\frac{(\mathbf{bc})^2}{BC}$	$\parallel BC$	$\rightarrow B$
4.	$\mathbf{f}^t_{cb}$ or $\mathbf{c}_b\mathbf{c}_1$	-	$\perp BC$ or $\mathbf{b}_1\mathbf{c}_b$	-
5.	$\mathbf{f}^c_{cd}$ or $\mathbf{d}_1\mathbf{c}_d$	$\frac{(\mathbf{dc})^2}{DC}$	$\parallel DC$	$\rightarrow D$
6.	$\mathbf{f}^t_{cd}$ or $\mathbf{c}_d\mathbf{c}_1$	-	$\perp DC$ or $\mathbf{d}_1\mathbf{c}_d$	-

Construct the acceleration diagram as follows:

(a) Select the pole point $\mathbf{a}_1$ or $\mathbf{d}_1$.

(b) Take the first vector from the above table, i.e., take $\mathbf{a}_1\mathbf{b}_a$ to a convenient scale in the proper direction and sense.

(c) Add the second vector to the first and then the third vector to the second.

(d) For the addition of the fourth vector, draw a line perpendicular to BC through the head $\mathbf{c}_b$ of the third vector. The magnitude of the fourth vector is unknown and $\mathbf{c}_1$ can lie on either side of $\mathbf{c}_b$.

(e) Take the fifth vector from $\mathbf{d}_1$.

(f) For the addition of the sixth vector to the fifth, draw a line perpendicular to DC through the head $\mathbf{c}_d$ of the fifth vector.

The intersection of this line with the line drawn in the step (d) locates the point $\mathbf{c}_1$.

Total acceleration of B = $\mathbf{a}_1\ \mathbf{b}_1$

Total acceleration of C relative to B = $\mathbf{b_1 c_1}$
Total acceleration of C = $\mathbf{d_1 c_1}$

Angular Acceleration of Links

From the foregoing discussion, it can be observed that the tangential component of acceleration occurs due to the angular acceleration of a link.

Tangential acc. of B rel. to A,

$$\mathbf{f^t_{ba}} = \alpha.AB = \alpha.BA$$

where α = angular acceleration of the link AB

Thus, angular acceleration of a link can be found if the tangential acceleration is known.

Referring to Fig. 3.2,

Tangential acc. of C rel. to B, $\mathbf{f^t_{cb}} = \mathbf{c_b\,c_1}$

i.e., acceleration of C relative to P is in a direction $\mathbf{c_b}$ to $\mathbf{c_1}$ or in a counter-clockwise direction about B.

As $f^t_{cb} = \alpha_{cb} CB$

$\therefore$ $\alpha_{cb} = f^t_{cb}/CB$

Tangential acc. of B rel. to C, $f^t_{bc} = \mathbf{c_1\,c_b}$

i.e., acceleration of B relative to C is in a direction $\mathbf{c_1}$ to $\mathbf{c_b}$ or in counter-clockwise direction about C with magnitude, $\alpha_{bc} = f^t_{bc}\,/\,BC$ which is the same as α_{cb}.

Thus, angular acceleration of a link about one extremity is the same in magnitude and direction as the angular acceleration about the other.

Tangential acc. of C rel. to D, $\mathbf{f^t_{cd}} = \mathbf{c_d\,c_1}$

i.e., C relative to D moves in a direction from $\mathbf{c_d}$ to $\mathbf{c_1}$ or C moves in the counter-clockwise direction about D.

$$\alpha_{cd} = \frac{f^t_{cd}}{CD} = \frac{c_d c_1}{CD}$$

ACCELERATION OF INTERMEDIATE AND OFFSET POINTS

Intermediate Point

The acceleration of intermediate points on the links can be obtained by dividing the acceleration vectors in the same ratio as the points divide the links. For point E on the link BC (Fig. 3.2),

$$\frac{BE}{BC} = \frac{\mathbf{b_1\,e_1}}{\mathbf{b_1\,c_1}}$$

$\mathbf{a_1\,e_1}$ gives the total acceleration of the point E.

Offset Points

The acceleration of an offset point on a link, such as F on BC (Fig. 3.2), can be determined by applying any of the following methods:

1. Writing the vector equation,

$$\mathbf{f}_{fb} + \mathbf{f}_{ba} = \mathbf{f}_{fc} + \mathbf{f}_{cd}$$

or $$\mathbf{f}_{ba} + \mathbf{f}_{fb} = \mathbf{f}_{cd} + \mathbf{f}_{fc}$$

or $$\mathbf{f}_{ba} + \mathbf{f}^{c}_{fb} + \mathbf{f}^{t}_{fb} = \mathbf{f}_{cd} + \mathbf{f}^{c}_{fc} + \mathbf{f}^{c}_{fc}$$

or $$\mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{f}_b + \mathbf{f}_b\mathbf{f}_1 = \mathbf{d}_1\mathbf{c}_1 + \mathbf{f}_1\mathbf{f}_c + \mathbf{f}_c\mathbf{f}_1$$

The equation can be easily solved graphically as shown in Fig. 3.2(d). $\mathbf{a}_1\mathbf{f}_1$ represents the acceleration of F relative to A or D.

2. Writing the vector equation,

$$\mathbf{f}_{fa} = \mathbf{f}_{fb} + \mathbf{f}_{ba}$$

$$= \mathbf{f}_{ba} + \mathbf{f}_{fb}$$

$$= \mathbf{f}_{ba} + \mathbf{f}^{c}_{fb} + \mathbf{f}^{t}_{fb}$$

or $$\mathbf{a}_1\mathbf{f}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{f}_b + \mathbf{f}_b\mathbf{f}_1$$

$\mathbf{f}_{ba}$ already exists on the acceleration diagram.

$$\mathbf{f}^{c}_{fb} = \frac{(bf)^2}{BF}, \parallel FB, \text{ direction towards } B.$$

$$\mathbf{f}^{t}_{fb} = \alpha_{fb} \times FB = \alpha_{cb} \times FB$$

$$= \frac{\mathbf{f}^{t}_{cb}}{CB} \times FB \perp \text{ to } FB; \text{ direction } \mathbf{b} \text{ to } \mathbf{f}$$

$\alpha_{fb} = \alpha_{cb}$, because angular acceleration of all the points on the link BCF about the point B is the same (counter-clockwise).

$\mathbf{f}_{fa}$ can be found in this way.

3. ***By acceleration image method*** In the previous chapter, it was mentioned that velocity images are useful in finding the velocities of offset points of links. In the same way, *acceleration images* are also helpful to find the accelerations of offset points of the links. The acceleration image of a link is obtained in the same manner as a velocity image. It can be proved that the triangle $\mathbf{b}_1\mathbf{c}_1\mathbf{f}_1$ is similar to the triangle BCF in Figs 3.2(d) and (a).

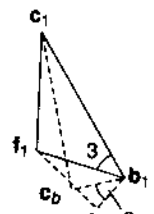


Fig. 3.3

Let ω' = angular velocity of the link BCF

α = angular acceleration of the link BCF

Referring to Figs 3.2(a) and 3.3,

$$\frac{\mathbf{b}_1\,\mathbf{f}_b}{\mathbf{b}_1\,\mathbf{c}_b} = \frac{\omega'^2 BF}{\omega'^2 BC} = \frac{BF}{BC} = \frac{\alpha BF}{\alpha BC} = \frac{\mathbf{f}_b\,\mathbf{f}_1}{\mathbf{c}_b\,\mathbf{c}_1}$$

$\mathbf{b}_1\mathbf{f}_b\mathbf{f}_1$ and $\mathbf{b}_1\mathbf{c}_b\mathbf{c}_1$ are two right-angled triangles in which the ratio of the two corresponding sides is the same as proved above. Therefore, the two triangles are similar.

$$\frac{\mathbf{b}_1\,\mathbf{f}_1}{\mathbf{b}_1\,\mathbf{c}_1} = \frac{BF}{BC} = k$$

Also, $\angle \mathbf{f}_b\,\mathbf{b}_1\,\mathbf{f}_1 = \angle \mathbf{c}_b\,\mathbf{b}_1\,\mathbf{c}_1$

or $$\angle \mathbf{f}_b\mathbf{b}_1\mathbf{f}_1 = \angle \mathbf{c}_b\mathbf{b}_1\mathbf{f}_1 = \angle \mathbf{c}_b\mathbf{b}_1\mathbf{c}_1 - \angle \mathbf{c}_b\mathbf{b}_1\mathbf{f}_1$$

or $$\angle 3 = \angle 2 = \angle 1 \quad (\because \mathbf{b}_1\mathbf{f}_b \parallel BF, \mathbf{b}_1\mathbf{c}_b \parallel BC)$$

Now, in Δs $\mathbf{b_1f_1c_1}$ and BFC,

$$\angle 3 = \angle 1$$

and
$$\frac{\mathbf{b_1 f_1}}{\mathbf{b_1 c_1}} = \frac{BF}{BC} = k$$

Therefore, the two triangles are similar.

Thus, to find the acceleration of an offset point on a link, a triangle similar to the one formed in the configuration diagram can be made on the acceleration image of the link in such a way that the sequence of letters is the same, i.e., $\mathbf{b_1f_1c_1}$ is clockwise, so should be BFC.

An easier method of making the triangle $\mathbf{b_1f_1}$ similar to BFC is by marking BC' on BC equal to $\mathbf{b_1c_1}$ and drawing a line parallel to CF, meeting BF in F'. $BC'F'$ is the exact size of the triangle to be made on $\mathbf{b_1c_1}$. Take $\mathbf{b_1f_1} = BF'$ and $\mathbf{c_1f_1} = C'F'$.

Thus, the point $\mathbf{f_1}$ is obtained.

3.4 SLIDER-CRANK MECHANISM

The configuration and the velocity diagrams of a slider-crank mechanism discussed in Sec. 2.8 have been reproduced in Figs. 3.4(a) and (b).

Writing the acceleration equation,

Acc. of B rel. to O = Acc. of B rel. to A + Acc. of A rel. to O

$$\mathbf{f}_{bo} = \mathbf{f}_{ba} + \mathbf{f}_{ao}$$

$$\mathbf{f}_{bg} = \mathbf{f}_{ao} + \mathbf{f}_{ba} = \mathbf{f}_{ao} + \mathbf{f}^c_{ba} + \mathbf{f}^t_{ba}$$

$$\mathbf{g_1 b_1} = \mathbf{o_1 a_1} + \mathbf{a_1 b_a} + \mathbf{b_a b_1}$$

The crank OA rotates at a uniform velocity, therefore, the acceleration of A relative to O has only the centripetal component. Similarly, the slider moves in a linear direction and thus has no centripetal component.

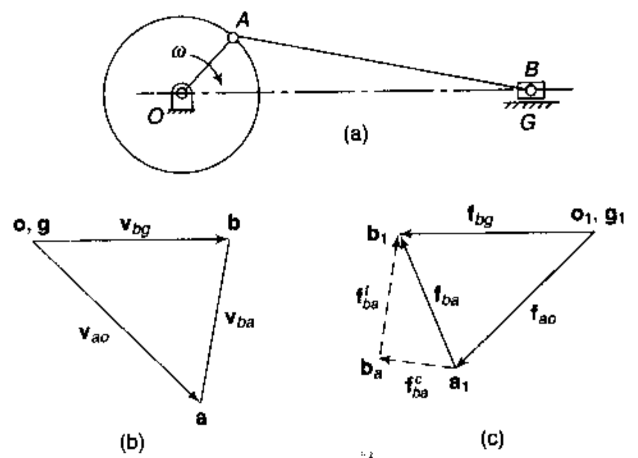


Fig. 3.4

Setting the vector table:

SN	Vector	Magnitude	Direction	Sense
1.	$\mathbf{f}_{ao}$ or $\mathbf{o_1a_1}$	$\frac{(\mathbf{oa})^2}{OA}$	$\parallel OA$	$\rightarrow$ O
2.	$\mathbf{f}^c_{ba}$ or $\mathbf{a_1 b_a}$	$\frac{(\mathbf{ab})^2}{AB}$	$\parallel AB$	$\rightarrow$ A
3.	$\mathbf{f}^t_{ba}$ or $\mathbf{b_a b_1}$	-	$\perp AB$	-
4.	$\mathbf{f}_{bg}$ or $\mathbf{g_1b_1}$	-	$\parallel$ to line of motion of B	-

Construct the acceleration diagram as follows:

1. Take the first vector $\mathbf{f}_{ao}$.
2. Add the second vector to the first.
3. For the third vector, draw a line ⊥ to AB through the head $\mathbf{b}_a$ of the second vector.
4. For the fourth vector, draw a line through $\mathbf{g}_1$ parallel to the line of motion of the slider.

   This completes the velocity diagram.

   Acceleration of the slider $B = \mathbf{o}_1\,\mathbf{b}_1$ (or $\mathbf{g}_1\,\mathbf{b}_1$)

   Total acceleration of B relative to $A = \mathbf{a}_1\,\mathbf{b}_1$

Note that for the given configuration of the mechanism, the direction of the acceleration of the slider is opposite to that of the velocity. Therefore, the acceleration is negative or the slider decelerates while moving to the right.

Example 3.1 *Figure 3.5(a) shows the configuration diagram of a four-link mechanism along with the lengths of the links in mm. The link AB has an instantaneous angular velocity of 10.5 rad/s and a retardation of 26 rad/s² in the counter-clockwise direction. Find*

(i) the angular accelerations of the links BC and CD

(ii) the linear accelerations of the points E, F and G

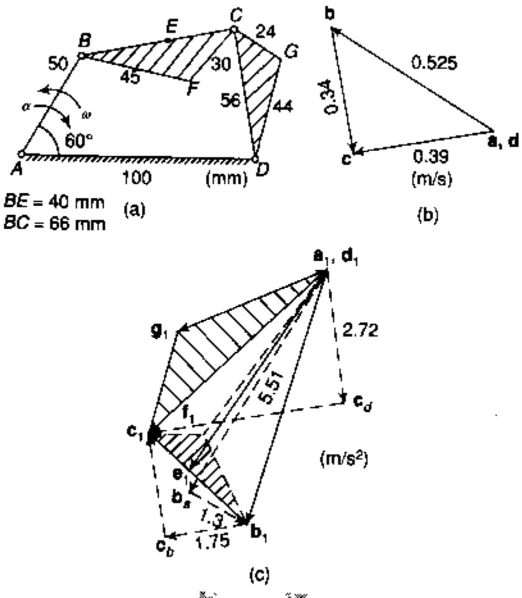


Fig. 3.5

Solution $v_b = 10.5 \times 0.05 = 0.525$ m/s

Complete the velocity diagram [Fig. 3.5(b)] as explained in Example 2.1.

Writing the vector equation for acceleration,

Acc. of C rel. to A = Acc. of C rel. to B + Acc. of B rel. to A

$$\mathbf{f}_{ca} = \mathbf{f}_{cb} + \mathbf{f}_{ba}$$

or $$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

or $$\mathbf{d}_1\,\mathbf{c}_1 = \mathbf{a}_1\,\mathbf{b}_1 + \mathbf{b}_1\,\mathbf{c}_1$$

Each vector has a centripetal and a tangential component,

$$\therefore \quad \mathbf{f}^c_{cd} + \mathbf{f}^t_{cd} = \mathbf{f}^c_{ba} + \mathbf{f}^t_{ba} + \mathbf{f}^c_{cb} + \mathbf{f}^t_{cb}$$

or $$\mathbf{d}_1\,\mathbf{c}_d + \mathbf{c}_d\,\mathbf{c}_1 = \mathbf{b}_a + \mathbf{b}_a\,\mathbf{b}_1 + \mathbf{b}_1\,\mathbf{c}_b + \mathbf{c}_b\,\mathbf{c}_1$$

Set the vector table (Table 1) on the next page.

Draw the acceleration diagram as follows:

(i) Take the pole point $\mathbf{a}_1$ or $\mathbf{d}_1$ [Fig. 3.5(c)].

(ii) Starting from $\mathbf{a}_1$, take the first vector $\mathbf{a}_1\,\mathbf{b}_a$.

(iii) To the first vector, add the second vector and to the second vector, add the third.

(iv) The vector 4 is known in direction only. Therefore, through the head $\mathbf{c}_b$ of the third vector, draw a line, ⊥ to BC. The point $\mathbf{c}_1$ of the fourth vector is to lie on this line.

(v) Start with $\mathbf{d}_1$ and take the fifth vector $\mathbf{d}_1\,\mathbf{c}_d$.

Table 1

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^c_{ba}$ or $\mathbf{a}_1\mathbf{b}_a$	$\frac{(\mathbf{ab})^2}{AB}=\frac{(0.525)^2}{0.05}=5.51$	$\parallel AB$	$\to A$
2.	$\mathbf{f}^t_{ba}$ or $\mathbf{b}_a\mathbf{b}_1$	$\alpha \times AB = 26 \times 0.05 = 1.3$	$\perp AB$ or $\parallel$ **ab**	$\to$ **a**
3.	$\mathbf{f}^t_{cb}$ or $\mathbf{b}_1\mathbf{c}_b$	$\frac{(\mathbf{bc})^2}{BC}=\frac{(0.34)^2}{0.066}=1.75$	$\parallel BC$	$\to B$
4.	$\mathbf{f}^t_{cb}$ or $\mathbf{b}_b\mathbf{c}_1$	-	$\perp B$	-
5.	$\mathbf{f}^c_{cd}$ or $\mathbf{d}_1\mathbf{c}_d$	$\frac{(\mathbf{dc})^2}{DC}=\frac{(0.39)^2}{0.56}=2.72$	$\parallel DC$	$\to D$
6.	$\mathbf{f}^t_{cd}$ or $\mathbf{c}_d\mathbf{c}_1$	-	$\perp B$	-

(vi) The sixth vector is known in direction only. Draw a line ⊥ to DC through head $\mathbf{c}_d$ of the fifth vector, the intersection of which with the line in the step (d) locates the point $\mathbf{c}_1$.

(vii) Join $\mathbf{a}_1\mathbf{b}_1$, $\mathbf{b}_1\mathbf{c}_1$ and $\mathbf{d}_1\mathbf{c}_1$.

Now, $\mathbf{a}_1\mathbf{b}_1$ represents the total accelerations of the point B relative to the point A.

Similarly, $\mathbf{b}_1\mathbf{c}_1$ is the total acceleration of C relative to B and $\mathbf{d}_1\mathbf{c}_1$ is the total acceleration of C relative to D.

Note In the acceleration diagram shown in Fig. 2.5c, the arrowhead has been put on the line joining points $\mathbf{b}_1$ and $\mathbf{c}_1$ in such a way that it represents the vector for acceleration of C relative to B. This satisfies the above equation. As the same equation

$$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

can also be put as

$$\mathbf{f}_{cd} + \mathbf{f}_{bc} = \mathbf{f}_{ba}$$

$$\mathbf{d}_1\mathbf{c}_1 + \mathbf{c}_1\mathbf{b}_1 = \mathbf{a}_1\mathbf{b}_1$$

This shows that on the same line joining $\mathbf{b}_1$ and $\mathbf{c}_1$, the arrowhead should be marked in the other direction so that the vector represents the acceleration of B relative to C to satisfy the latter equation.

This implies that in case both the ends of a link are in motion, the arrowhead may be put in either direction or no arrowhead is put at all. This is because every time it is not necessary to write the acceleration equation. The acceleration equation is helpful only at the initial stage for better comprehension.]

(i) *Angular accelerations*

$$\alpha_{bc} = \frac{f^t_{cb} \text{ or } \mathbf{c}_b\,\mathbf{c}_1}{BC} = \frac{2.25}{0.066} = \underline{34.09 \text{ rad/s}^2}$$

counter-clockwise

$$\alpha_{cd} = \frac{\mathbf{f}^t_{cd} \text{ or } \mathbf{c}_d\,\mathbf{c}_1}{CD} = \frac{4.43}{0.056}$$

$= \underline{79.11 \text{ rad/s}^2}$ counter-clockwise

(ii) *Linear accelerations*

(a) Locate point $\mathbf{e}_1$ on $\mathbf{b}_1\,\mathbf{c}_1$ such that

$$\frac{\mathbf{b}_1\mathbf{e}_1}{\mathbf{b}_1\mathbf{c}_1} = \frac{BE}{BC}$$

$$f_e = \mathbf{a}_1\mathbf{e}_1 = \underline{5.15 \text{ m/s}^2}$$

(b) Draw Δ $\mathbf{b}_1\mathbf{c}_1\mathbf{f}_1$ similar to Δ BCF keeping in mind that BCF as well as $\mathbf{b}_1\mathbf{c}_1\mathbf{f}_1$ are read in the same order (clockwise in this case).

$f_f = \mathbf{a}_1\mathbf{f}_1 = \underline{4.42 \text{ m/s}^2}$

(c) Linear acceleration of the point G can also be found by drawing the acceleration image of the triangle DCG on $\mathbf{d}_1\mathbf{c}_1$ in the acceleration diagram such that the order of the letters remains the same.

$f_g = \mathbf{d}_1\mathbf{g}_1 = \underline{3.9 \text{ m/s}^2}$

Example 3.2 *For the configuration of a slider-crank mechanism shown in Fig. 3.6(a), calculate the*

(i) *acceleration of the slider at B*
(ii) *acceleration of the point E*
(iii) *angular acceleration of the link AB*
OA rotates at 20 rad/s counter-clockwise.

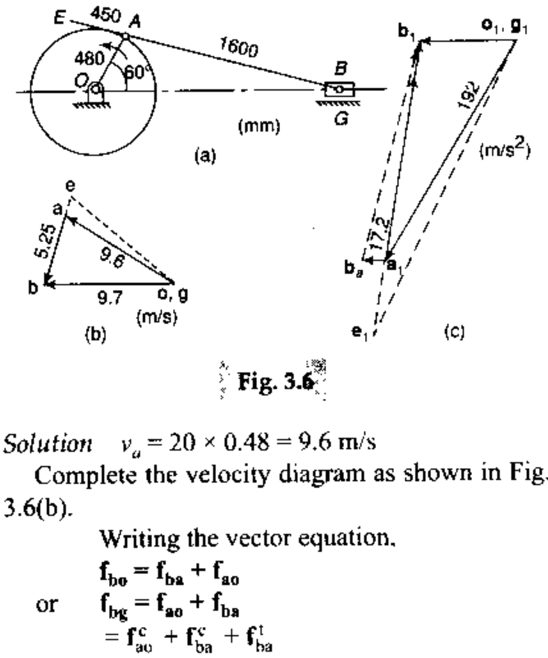


Fig. 3.6

Solution $v_a = 20 \times 0.48 = 9.6$ m/s

Complete the velocity diagram as shown in Fig. 3.6(b).

Writing the vector equation,

$\mathbf{f_{bo}} = \mathbf{f_{ba}} + \mathbf{f_{ao}}$

or $\mathbf{f_{bg}} = \mathbf{f_{ao}} + \mathbf{f_{ba}}$

$= \mathbf{f^c_{ao}} + \mathbf{f^c_{ba}} + \mathbf{f^t_{ba}}$

or $\mathbf{g_1\,b_1} = \mathbf{o_1\,a_1} + \mathbf{a_1\,b_a} + \mathbf{b_a\,b_1}$

Set the vector table (Table 2) as given below.

The acceleration diagram is drawn as follows:

(a) Take the pole point $\mathbf{o_1}$ or $\mathbf{g_1}$ [Fig. 3.6 (c)].

(b) Take the first vector $\mathbf{o_1a_1}$ and add the second vector.

(c) For the third vector, draw a line ⊥ to *AB* through the head $\mathbf{b_a}$ of the second vector.

(d) For the fourth vector, draw a line || to the line of motion of the slider through $\mathbf{g_1}$. The intersection of this line with the line drawn in the step (d) locates point $\mathbf{b_1}$.

(e) Join $\mathbf{a_1\,b_1}$.

(i) $f_b = \mathbf{g_1b_1} = 72$ m/s^2

As the direction of acceleration $\mathbf{f}_b$ is the same as of $\mathbf{v}_b$, this means the slider is accelerating at the instant.

(ii) Locate point $\mathbf{e_1}$ on $\mathbf{b_1a_1}$ produced such that

$$\frac{\mathbf{b_1\,e_1}}{\mathbf{b_1\,a_1}} = \frac{\mathbf{BE}}{\mathbf{BA}}$$

$f_e = \mathbf{o_1e_1} = \underline{236 \text{ m/s}^2}$

(iii) $$\alpha_{ab} = \frac{\mathbf{f}^t_{ba}}{AB} = \frac{\mathbf{b_a\,b_1}}{AB} = \frac{167}{1.6}$$

$= \underline{104 \text{ rad/s}^2}$ counter-clockwise

Example 3.3 *Figure 3.7(a) shows configuration of an engine mechanism. The dimensions are the following:*

Crank OA = 200 mm; Connecting rod AB = 600 mm; distance of centre of mass from crank end, AD = 200 mm. At the instant, the crank has an angular velocity of 50 rad/s clockwise and an angular acceleration of 800 rad/s². Calculate the

(i) *velocity of D and angular velocity of AB*
(ii) *acceleration of D and angular acceleration of AB*

Table 2

SN	Vector	Magnitude (m/s^2)	Direction	Sense
1.	$\mathbf{f^c_{ao}}$ or $\mathbf{o_1\,a_1}$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(9.6)^2}{0.48} = 192$	∥ *OA*	→ *O*
2.	$\mathbf{f^c_{ba}}$ or $\mathbf{a_1\,b_a}$	$\frac{(\mathbf{ab})^2}{AB} = \frac{(5.25)^2}{1.60} = 17.2$	∥ *AB*	→ *A*
3.	$\mathbf{f^t_{ba}}$ or $\mathbf{b_a\,b_1}$	-	⊥*AB*	-
4.	$\mathbf{f_{bg}}$ or $\mathbf{g_1\,b_1}$	-	∥ to slider motion	-

***(iii)* point on the connecting rod which has zero acceleration at this instant.**

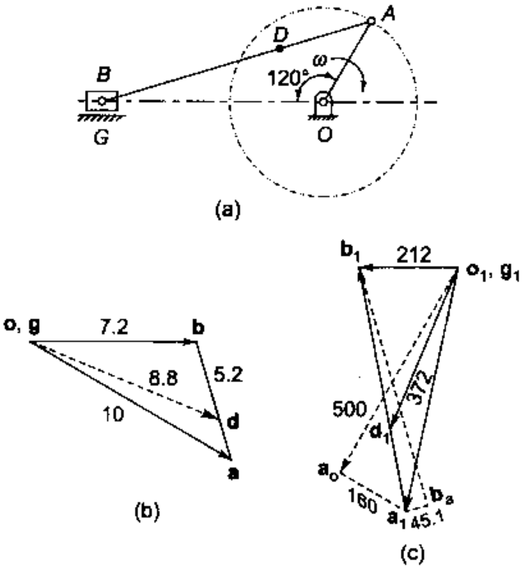


Fig. 3.7

Solution: $v_a = 50 \times 0.2 = 10$ m/s

Complete the velocity diagram as shown in Fig. 3.7 (b).

(i) Velocity of D = **od** = <u>8.8 m/s</u>

Angular velocity of AB = **ab**/AB = 5.2/ 0.6 = <u>8.67 rad/s.</u>

Writing the vector equation,

$$\mathbf{f_{bo}} = \mathbf{f_{ba}} + \mathbf{f_{ao}}$$

or $\quad \mathbf{f_{bg}} = \mathbf{f_{ao}} + \mathbf{f_{ba}}$

$$= \mathbf{f^c_{ao}} + \mathbf{f^t_{ao}} + \mathbf{f^c_{ba}} + \mathbf{f_t}$$

or $\quad \mathbf{g_1 b_1} = \mathbf{o_1 a_o} + \mathbf{a_o a_1} + \mathbf{a_1 b_a} + \mathbf{b_a b_1}$

Set the vector table (Table 3) as given below.

The acceleration diagram is drawn as follows:

(a) Take the pole point $\mathbf{o_1}$ or $\mathbf{g_1}$ [Fig. 3.7(c)].

(b) Take the first vector $\mathbf{o_1 a_1}$.

(c) Add the second vector to the first and then the third vector to the second.

(d) For the fourth vector, draw a line ⊥ to AB through the head $\mathbf{b_a}$ of the third vector.

(e) For the fifth vector, draw a line || to the line of motion of the slider through $\mathbf{g_1}$. The intersection of this line with the line drawn in step (d) locates point $\mathbf{b_1}$.

(f) Join $\mathbf{a_1}$ $\mathbf{b_1}$.

(ii) Locate point $\mathbf{d_1}$ on $\mathbf{b_1 a_1}$ produced such that

$$\frac{\mathbf{b_1 d_1}}{\mathbf{b_1 a_1}} = \frac{\mathbf{BD}}{\mathbf{BA}}$$

$f_d = \mathbf{o_1 d_1}$ = <u>372 m/s²</u>

$$\alpha_{ab} = \frac{\mathbf{f^t_{ba}}}{AB} = \frac{\mathbf{b_a b_1}}{AB} = \frac{521}{0.6}$$

= <u>868 rad/s²</u> clokwise

Table 3

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f^c_{ao}}$ or $\mathbf{o_1 a_o}$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(10)^2}{0.2} = 500$	‖ OA	→ O
2.	$\mathbf{f^t_{ao}}$ or $\mathbf{a_o a_1}$	$\alpha \times AB = 800 \times 0.2 = 160$	⊥ OA or ‖ **oa**	→ **a**
3.	$\mathbf{f^c_{ba}}$ or $\mathbf{a_1 b_a}$	$\frac{(\mathbf{ab})^2}{AB} = \frac{(5.2)^2}{0.6} = 45.1$	‖ AB	→ A
4.	$\mathbf{f^t_{ba}}$ or $\mathbf{b_a b_1}$	-	⊥ AB	-
5.	$\mathbf{f_{bg}}$ or $\mathbf{g_1 b_1}$	-	‖ to slider motion	-

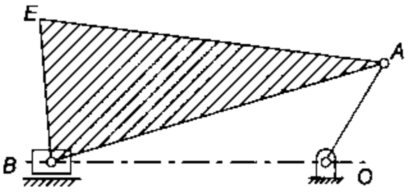


Fig. 3.8

To find the point on the connecting rod which has zero acceleration at this instant, draw triangle ABE on the configuration diagram similar to $\mathbf{a_1b_1o_1}$ such that the letters are in the same order, i.e., clockwise (Fig. 3.8). Then E is a point on the connecting rod with zero acceleration as it corresponds to zero acceleration of point O.

Example 3.4 ***In the mechanism shown in Fig. 3.9(a), the crank OA rotates at 210 rpm clockwise. For the given configuration, determine the acceleration of the slider D and angular acceleration of the link CD.***

Solution $v_a = \dfrac{2\pi \times 210}{60} \times 0.1 = 2.2$ m/s

Complete the velocity diagram as follows:

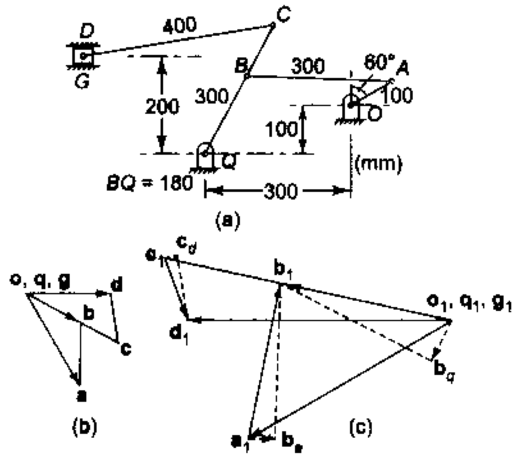


Fig. 3.9

- For the four-link mechanism $OABQ$, complete the velocity diagram as usual [Fig. 3.9(b)].
- Locate point **c** on vector **ob** extended so that $\dfrac{cq}{bq} = \dfrac{CQ}{BQ} = \dfrac{300}{180} = 1.667$
- Draw a horizontal line through **g** for the vector $\mathbf{v}_{dg}$ and a line $\perp CD$ for the vector $\mathbf{v}_{dc}$, the intersection of the two locates the point **d**. Thus the velocity diagram is completed.

Set the vector table (Table 4).

Table 4

S.N.	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^c_{ao}$ or $\mathbf{o}_1\,\mathbf{a}_1$	$\dfrac{(\mathbf{oa})^2}{OA} = \dfrac{(2.2)^2}{0.1} = 48.4$	$\parallel OA$	$\rightarrow O$
2.	$\mathbf{f}^c_{ba}$ or $\mathbf{a}_1\,\mathbf{b}_a$	$\dfrac{(\mathbf{ab})^2}{AB} = \dfrac{(1.29)^2}{0.3} = 5.55$	$\parallel AB$	$\rightarrow A$
3.	$\mathbf{f}^t_{ba}$ or $\mathbf{b}_a\,\mathbf{b}_1$	-	$\perp AB$	-
4.	$\mathbf{f}^c_{bq}$ or $\mathbf{q}_1\,\mathbf{b}_q$	$\dfrac{(\mathbf{bq})^2}{BQ} = \dfrac{(1.29)^2}{0.18} = 9.25$	$\parallel BQ$	-
5.	$\mathbf{f}^t_{bq}$ or $\mathbf{b}_q\,\mathbf{b}_1$	-	$\perp BQ$	-
6.	$\mathbf{f}^c_{dc}$ or $\mathbf{c}_1\,\mathbf{c}_d$	$\dfrac{(\mathbf{cd})^2}{CD} = \dfrac{(1.01)^2}{0.4} = 2.55$	$\parallel CD$	$\rightarrow C$
7.	$\mathbf{f}^t_{dc}$ or $\mathbf{c}_d\,\mathbf{d}_1$	-	$\perp CD$	-
8.	$\mathbf{f}_{dg}$ or $\mathbf{g}_1\,\mathbf{d}_1$	-	$\parallel$ to slider motion	-

The acceleration diagram is drawn as follows:

(i) From the pole point $\mathbf{o}_1$ take the first vector $\mathbf{o}_1\mathbf{a}_1$ [Fig. 3.9(c)].

(ii) Add the second vector by placing its tail at $\mathbf{a}_1$.

(iii) For the third vector $\mathbf{f}^t_{ba}$, draw a line $\perp AB$ through $\mathbf{b}_a$

(iv) Add the fourth vector by placing its tail at $\mathbf{q}_1$ and to add the fifth vector $\mathbf{f}^t_{bq}$, draw a line $\perp BQ$ through $\mathbf{b}_q$. Intersection of the two lines locates point $\mathbf{b}_1$.

(v) Locate point $\mathbf{c}_1$ on the vector $\mathbf{q}_1\mathbf{b}_1$ by extending it so that

$$\frac{\mathbf{c}_1\mathbf{q}_1}{\mathbf{b}_1\mathbf{q}_1} = \frac{CQ}{BQ} = \frac{300}{180} = 1.667.$$

(vi) Add the vector for centripetal acceleration $\mathbf{f}^c_{dc}$ of link CD by placing its tail at $\mathbf{c}_1$ and for its tangential component, draw a perpendicular line to it.

(vii) For the vector 8, draw a horizontal line through $\mathbf{g}$, the intersection of this line with the live drawn in (iii) locates point $\mathbf{d}_1$.

This completes the acceleration diagram.
Acceleration of slider $D = \mathbf{g}_1\mathbf{d}_1 = 54.4$ m/s
Angular acceleration of link CD,

$$\alpha_{cd} = \frac{\mathbf{f}^t_{cd} \text{ or } \mathbf{c}_d\mathbf{d}_1}{CD} = \frac{13.3}{0.4} = 33.25 \text{ rad/s}^2$$

Example 3.5 ***In the mechanism shown in Fig. 3.10(a), the crank OA rotates at 60 rpm. Determine the***

(i) linear acceleration of the slider at B

(ii) angular acceleration of the links AC, CQD and BD

Solution $v_a = \dfrac{2\pi \times 60}{60} \times 0.15 = 0.94$ m/s

Complete the velocity diagram as shown in Fig. 3.10(b).

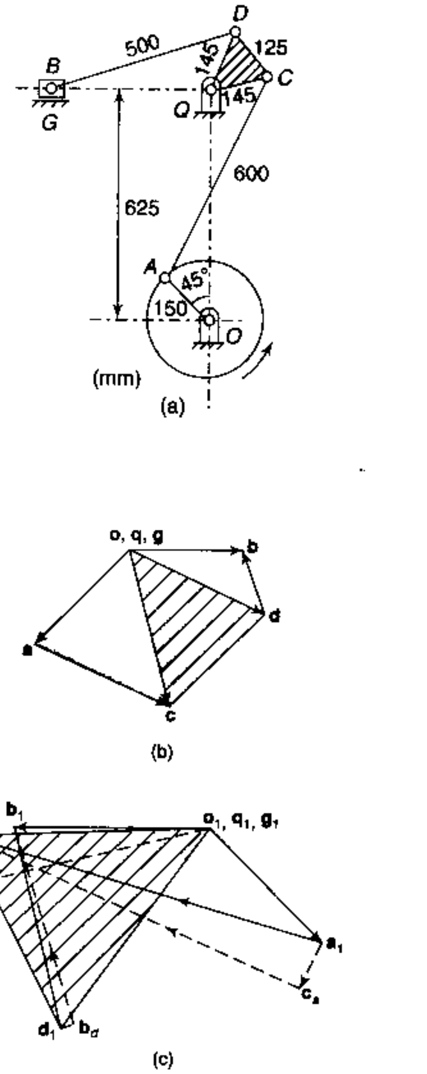


Fig. 3.10

It is a six-link mechanism. First, consider the four-link mechanism $OACQ$ and write the vector equation

$$\mathbf{f}_{co} = \mathbf{f}_{ca} + \mathbf{f}_{ao}$$

or $$\mathbf{f}_{cq} = \mathbf{f}_{ao} + \mathbf{f}_{ca}$$

or $$\mathbf{q}_1\mathbf{c}_1 = \mathbf{o}_1\mathbf{a}_1 + \mathbf{a}_1\mathbf{c}_1$$

Links AC and CQ each can have centripetal and tangential components.

$$\mathbf{f}_{cq}^{t} + \mathbf{f}_{cq}^{t} = \mathbf{f}_{ao}^{t} + \mathbf{f}_{ca}^{t} + \mathbf{f}_{ca}^{t}$$

or $$\mathbf{q}_1\,\mathbf{c}_q + \mathbf{c}_q\,\mathbf{c}_1 = \mathbf{o}_1\,\mathbf{a}_1 + \mathbf{a}_1\,\mathbf{c}_a + \mathbf{c}_a\,\mathbf{a}_1$$

Set the following vector table (Table 5).

Complete the acceleration vector diagram $\mathbf{o}_1\mathbf{a}_1\mathbf{c}_1\mathbf{q}_1$ as usual [Fig. 3.10(c)].

Draw $\Delta \mathbf{c}_1\mathbf{q}_1\mathbf{d}_1$ similar to ΔCQD such that both are read in the same sense, i.e., clockwise.

Write the vector equation for the slider-crank mechanism QDB,

$$\mathbf{f}_{bq} = \mathbf{f}_{bd} + \mathbf{f}_{dq}$$

or $$\mathbf{f}_{bg} = \mathbf{f}_{dq} + \mathbf{f}_{bd}$$

or $$\mathbf{g}_1\mathbf{b}_1 = \mathbf{q}_1\mathbf{d}_1 + \mathbf{d}_1\mathbf{b}_1$$

From this equation $\mathbf{q}_1\mathbf{d}_1$ is already drawn in the diagram and $\mathbf{g}_1\mathbf{b}_1$ is a linear acceleration component.

$$\mathbf{f}_{bg} = \mathbf{f}_{dq} + \mathbf{f}_{bd}^{c} + \mathbf{f}_{bd}^{t}$$

or $$\mathbf{g}_1\mathbf{b}_1 = \mathbf{q}_1\mathbf{d}_1 + \mathbf{d}_1\mathbf{b}_d + \mathbf{b}_d\mathbf{b}_1$$

Set the following vector table (Table 6).

Complete the acceleration vector diagram $\mathbf{q}_1\,\mathbf{d}_1\,\mathbf{b}_1\,\mathbf{g}_1$.

(i) $f_g = \mathbf{g}_1\,\mathbf{b}_1 = 7\text{ m/s}^2$ towards left

As the acceleration $\mathbf{f}_b$ is opposite to $\mathbf{v}_b$, the slider is decelerating.

(ii) $$\alpha_{ac} = \frac{\mathbf{f}_{ca}^{t} \text{ or } \mathbf{c}_a\mathbf{c}_1}{AC} = \frac{13.8}{0.6} = \underline{23\text{ rad/s}^2}$$ counter-clockwise

Table 5

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}_{ao}$ or $\mathbf{o}_1\,\mathbf{a}_1$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(0.94)^2}{0.15} = 5.92$	$\parallel OA$	$\to O$
2.	$\mathbf{f}_{ca}^{c}$ or $\mathbf{a}_1\mathbf{c}_a$	$\frac{(\mathbf{ac})^2}{AC} = \frac{(1.035)^2}{0.60} = 1.79$	$\parallel AC$	$\to A$
3.	$\mathbf{f}_{ca}^{t}$ or $\mathbf{c}_a\mathbf{c}_1$	-	$\perp AC$	-
4.	$\mathbf{f}_{cq}^{c}$ or $\mathbf{q}_1\mathbf{c}_q$	$\frac{(\mathbf{qc})^2}{QC} = \frac{(1.14)^2}{0.145} = 8.96$	$\parallel QC$	$\to Q$
5.	$\mathbf{f}_{cq}^{t}$ or $\mathbf{c}_q\mathbf{c}_1$	-	$\perp QC$	-

Table 6

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}_{dq}$ or $\mathbf{q}_1\,\mathbf{d}_1$	Already drawn	-	-
2.	$\mathbf{f}_{bd}^{c}$ or $\mathbf{d}_1\mathbf{b}_d$	$\frac{(\mathbf{db})^2}{DB} = \frac{(0.495)^2}{0.50} = 0.49$	$\parallel DB$	$\to D$
3.	$\mathbf{f}_{bd}^{t}$ or $\mathbf{b}_d\,\mathbf{b}_1$	-	$\perp DB$	-
4.	$\mathbf{f}_{bg}$ or $\mathbf{g}_1\,\mathbf{b}_1$	-	$\parallel$ to slider motion	-

$$\alpha_{cqd} = \frac{\mathbf{f}^t_{cq} \text{ or } \mathbf{c}_q\mathbf{c}_1}{QC} = \frac{2.0}{0.145} = \underline{13.8 \text{ rad/s}^2}$$

counter-clockwise

$$\alpha_{bd} = \frac{\mathbf{f}^t_{bd} \text{ or } \mathbf{b}_d\mathbf{b}_1}{BD} = \frac{7.2}{0.5} = \underline{14.4 \text{ rad/s}^2}$$

clockwise

Example 3.6 *In the mechanism shown in Fig. 3.11(a), the crank OA rotates at 210 rpm clockwise. For the given configuration, determine the velocities and accelerations of the sliders B, D and F.*

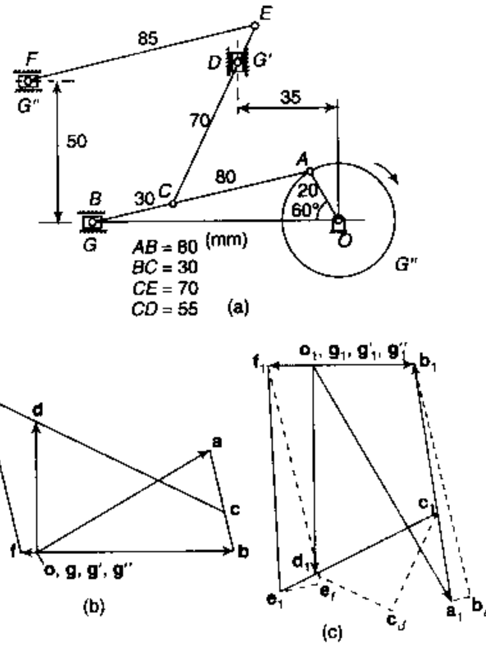


Fig. 3.11

Solution $v_a = \frac{2\pi \times 210}{60} \times 0.2 = 4.4$ m/s

Complete the velocity diagram as follows [Fig. 3.11(b)]:

- For the slider-crank mechanism *OAB*, complete the velocity diagram as usual.
- Locate the point **c** on the vector **ab**.
- Draw a vertical line through **g′** for the vector $\mathbf{v}_{dg}'$ and a line $\perp CD$ for the vector $\mathbf{v}_{dc}$, the intersection of the two locates the point **d**.
- Extend the vector **cd** to **e** such that **ce/cd** = *CE/CD*.
- Draw a horizontal line through **g″** for the vector $\mathbf{v}_{fg}''$ and a line $\perp EF$ for the vector $\mathbf{v}_{fe}$, the intersection of the two locates the point **f**.

Thus, the velocity diagram is completed.

Velocity of slider B = **gb** = $\underline{4.65 \text{ m/s}}$

Velocity of slider D = **g′d** = $\underline{2.85 \text{ m/s}}$

Velocity of slider F = **g″f** = $\underline{0.35 \text{ m/s}}$

Set the vector table (Table 7) as shown in the following page:

The acceleration diagram is drawn as follows:

(i) From the pole point $\mathbf{o}_1$, take the first vector $\mathbf{o}_1\mathbf{a}_1$ [Fig. 3.11(c)].

(ii) Add the second vector by placing its tail at $\mathbf{b}_1$.

(iii) For the third vector $\mathbf{f}^t_{ba}$, draw a line $\perp AB$ through $\mathbf{b}_1$ and for the fourth vector a horizontal line through **g**, the intersection of the two lines locates point $\mathbf{b}_1$.

(iv) Locate point $\mathbf{c}_1$ on the vector $\mathbf{a}_1\mathbf{b}_1$.

(v) Add the vector for centripetal acceleration $\mathbf{f}^c_{dc}$ of link *CD* and for its tangential component, draw a perpendicular line to it.

(vi) For the vector 7, draw a vertical line through **g′**, the intersection of this line to the previous line locates the point $\mathbf{d}_1$.

(vii) Join $\mathbf{c}_1\mathbf{d}_1$ and locate point $\mathbf{e}_1$ on its extension.

(viii) Take the vector 8 and draw line $\mathbf{e}_1\mathbf{e}_f$ parallel to *EF* and draw a line for the tangential component.

(ix) For the vector 10, take a horizontal line through $\mathbf{g}_1''$ and the intersection of this with the previous line locates the point $\mathbf{f}_1$.

This completes the acceleration diagram.

Acceleration of slider B = $\mathbf{g}_1\mathbf{b}_1$ = $\underline{36 \text{ m/s}^2}$

Acceleration of slider D = $\mathbf{g'}_1\mathbf{d}_1$ = $\underline{74 \text{ m/s}}$

Acceleration of slider F = $\mathbf{g''}_1\mathbf{f}_1$ = $\underline{16 \text{ m/s}}$

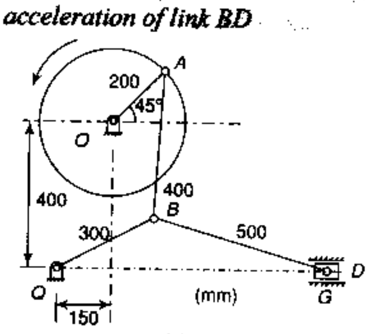
Table 7

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	f^c_{ao} or o_1a_1	$\frac{(oa)^2}{OA}=\frac{(4.4)^2}{0.2}=96.8$	$\parallel OA$	$\to O$
2.	f^c_{ba} or a_1b_a	$\frac{(ab)^2}{AB}=\frac{(2.26)^2}{0.8}=6.4$	$\parallel AB$	$\to A$
3.	f^t_{ba} or b_ab_1	-	$\perp AB$	-
4.	f_{bg} or g_1b_1		$\parallel$ to slider motion	-
5.	f^c_{dc} or c_1c_d	$\frac{(cd)^2}{CD}=\frac{(4.58)^2}{0.55}=38.1$	$\parallel CD$	$\to C$
6.	f^t_{dc} or c_dd_1	-	$\perp CD$	-
7.	f_{dg}' or $g_1'd_1$	-	$\parallel$ to slider motion	-
8.	f^c_{fe} or e_1e_f	$\frac{(ef)^2}{EF}=\frac{(3.49)^2}{0.85}=14.3$	[illegible]	$\to A$
9.	f^c_{fe} or $e_f\ f_1$	-	$\perp EF$	-
10.	$f_{fg'}$ or $g_1'f_1$	-	$\parallel$ to slider motion	-

Example 3.7 *In the toggle meehanism shown in Fig. 3.12(a), the crank OA rotates at 210 rpm counter-clockwise increasing at the rate of 60 rad/s². For the given configuration, determine*

(a) velocity of slider D and the angular velocity of link BD

(b) acceleration of slider D and the angular acceleration of link BD



(a)

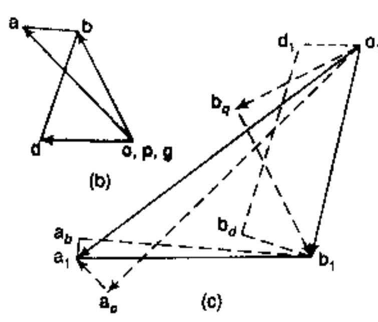


Fig. 3.12

Solution $v_a = \frac{2\pi \times 210}{60} \times 0.2 = 4.4$ m/s

Complete the velocity diagram as follows [Fig. 3.12(b)]:

- Take the vector **oa** representing $\mathbf{v}_a$.
- Draw lines **ab** $\perp AB$ through **a** and **qb**

$\perp QB$ through **q**, the intersection locates the point **b**.

- Draw the line **bd** $\perp BD$ through **b** and a horizontal line through **q** or **g** to represent the line of motion of the slider D. The intersection of the two lines locates the point **d**.

Velocity of slider D = **gd** = 2.54 m/s

Angular velocity of BD = **bd**/BD = 3.16/ 0.5 = 6.32 rad/s.

Set the following vector table (Table 8):

For the acceleration diagram, adopt the following steps:

(i) Take the pole point $\mathbf{o}_1$ or $\mathbf{c}_1$ [Fig. 3.12(c)].

(ii) Starting from $\mathbf{o}_1$, take the first vector $\mathbf{o}_1\mathbf{a}_o$. To the first vector, add the second vector. Thus, the total acceleration $\mathbf{o}_1\mathbf{a}_1$ of A relative to O is obtained.

(iii) Take the third vector and place its tail at $\mathbf{q}_1$ and through its head draw a perpendicular line to have the fourth vector.

(iv) Take the fifth vector and place its tail at $\mathbf{a}_1$. Through its head draw a perpendicular line to add the sixth vector.

(v) The intersection of lines of the fourth and sixth vectors locates the point $\mathbf{b}_1$.

(vi) Take the seventh vector and put its tail at $\mathbf{b}_1$. Through its head, draw a perpendicular line to add the eighth vector.

(vii) For the ninth vector, draw a line through $\mathbf{g}_1$ parallel to the slider motion.

(viii) The intersection of lines of the eighth and ninth vectors locates the point $\mathbf{d}_1$.

Acceleration of slider D = $\mathbf{g}_1\mathbf{d}_1$ = 16.4 m/s²

Angular acceleration of BD = $\mathbf{b}_d\mathbf{d}_1/BD$ = 5.46/0.5 = 109.2 rad/s².

Table 8

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	f^c_{ao} or $\mathbf{o}_1\mathbf{a}_o$	$\frac{(oa)^2}{OA} = \frac{(4.4)^2}{0.2} = 96.8$	$\parallel OA$	$\to O$
2.	f^t_{ao} or $\mathbf{a}_o\mathbf{a}_1$	$\alpha \times OA = 60 \times 0.2 = 12$	$\perp OA$	-
3.	f^c_{bq} or $\mathbf{q}_1\mathbf{b}_q$	$\frac{(bq)^2}{BQ} = \frac{(3.39)^2}{0.3} = 38.3$	$\parallel BQ$	$\to Q$
4.	f^t_{bq} or $\mathbf{b}_q\mathbf{b}_1$	-	$\perp BQ$	-
5.	f^c_{ba} or $\mathbf{a}_1\mathbf{a}_b$	$\frac{(ab)^2}{AB} = \frac{(1.54)^2}{0.4} = 5.93$	$\parallel AB$	$\to A$
6.	f^t_{ba} or $\mathbf{a}_b\mathbf{a}_1$	-	$\perp AB$	-
7.	f^c_{db} or $\mathbf{b}_1\mathbf{b}_d$	$\frac{(bd)^2}{BD} = \frac{(3.16)^2}{0.5} = 20$	$\parallel BD$	$\to B$
8.	f^t_{db} or $\mathbf{b}_1\mathbf{d}_1$	-	$\perp BD$	-
9.	f^t_{dg} or $\mathbf{g}_1\mathbf{d}_1$	-	$\parallel$ to slider motion	-

Example 3.8 *An Andrew variable-stroke engine mechanism is shown in Fig. 3.13(a). The crank OA rotates at 100 rpm. Find the*

(i) linear acceleration of the slider at D

(ii) angular acceleration of the links AC, BC and CD.

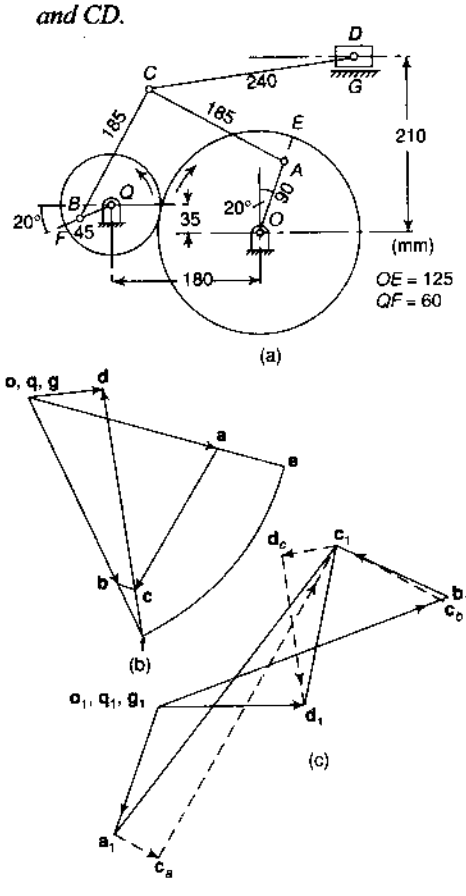


Fig. 3.13

Solution $v_a = \dfrac{2\pi \times 100}{60} \times 0.09 = 0.94$ m/s

Complete the velocity diagram as shown in Fig. 3.13(b). The procedure is explained in Example 2.10.

Write the acceleration vector equation noting that the cranks OA and QB rotate at different uniform speeds.

For the linkage $OACBQ$,

$\mathbf{f}_{ca} + \mathbf{f}_{ao} = \mathbf{f}_{cb} + \mathbf{f}_{bq}$ or $\mathbf{f}_{ao} + \mathbf{f}_{ca} = \mathbf{f}_{bq} + \mathbf{f}_{cb}$

$\mathbf{o_1 a_1} + \mathbf{a_1 c_1} = \mathbf{q_1 b_1} + \mathbf{b_1 c_1}$

Links AC and BC each have two components,

$\mathbf{f}_{ao} + \mathbf{f}^c_{ca} + \mathbf{f}^t_{ca} = \mathbf{f}_{bq} + \mathbf{f}^c_{cb} + \mathbf{f}^t_{cb}$

or $\mathbf{o_1 a_1} + \mathbf{a_1 c_a} + \mathbf{c_a c_1} = \mathbf{q_1 b_1} + \mathbf{b_1 c_b} + \mathbf{c_b c_1}$

Set the following vector table (Table 9).

Draw the acceleration diagram as follows:

(a) From the pole point $\mathbf{o_1}$, take the first vector and add the second vector to it as shown in Fig. 3.13 (c).

(b) Through the head $\mathbf{c_a}$ of the second vector, draw a line ⊥ to AC for the third vector.

(c) From $\mathbf{q_1}$ (or $\mathbf{o_1}$), take the fourth vector and add the fifth vector to it.

(d) Through the head $\mathbf{c_b}$ of the fifth vector, draw a line ⊥ to BC for the sixth vector.

The intersection of the lines drawn in steps (b) and (d) locates the point $\mathbf{c_1}$.

Now,

$\mathbf{f}_{do} = \mathbf{f}_{dc} + \mathbf{f}_{co}$ or $\mathbf{f}_{dg} = \mathbf{f}_{co} + \mathbf{f}_{dc}$

Since $\mathbf{f}_{dc}$ has two components,

$\mathbf{f}_{dg} = \mathbf{f}_{co} + \mathbf{f}^c_{dc} + \mathbf{f}^t_{dc}$

or $\mathbf{g_1 d_1} = \mathbf{o_1 c_1} + \mathbf{c_1 d_c} + \mathbf{d_c d_1}$

Set the following vector table (Table 10).

From $\mathbf{c_1}$ draw the second vector and draw a line ⊥ to CD through the head of the second vector. Draw a line parallel to the line of motion of the slider through $\mathbf{g_1}$. Thus, the point $\mathbf{d_1}$ is located.

(i) $f_d = \mathbf{o_1 d_1} = \underline{10.65 \text{ m/s}^2}$

(ii) $\alpha_{ac} = \dfrac{\mathbf{f}^t_{ca} \text{ or } \mathbf{c_a c_1}}{AC} = \dfrac{26.4}{0.185} = \underline{142.7 \text{ rad/s}^2}$ clockwise

$\alpha_{bc} = \dfrac{\mathbf{f}^t_{cb} \text{ or } \mathbf{c_b c_1}}{BC} = \dfrac{8.85}{0.185} = \underline{47.8 \text{ rad/s}^2}$ counter-clockwise

$\alpha_{cd} = \dfrac{\mathbf{f}^t_{dc} \text{ or } \mathbf{d_c d_1}}{CD} = \dfrac{11.2}{0.24} = \underline{46.7 \text{ rad/s}^2}$ clockwise

Table 9

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}_{ao}$ or $\mathbf{o}_1\,\mathbf{a}_1$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(0.94)^2}{0.09} = 9.87$	$\parallel OA$	$\rightarrow O$
2.	$\mathbf{f}^c_{ca}$ or $\mathbf{a}_1\,\mathbf{c}_a$	$\frac{(\mathbf{ac})^2}{AC} = \frac{(0.81)^2}{0.185} = 3.55$	$\parallel AC$	$\rightarrow A$
3.	$\mathbf{f}^t_{ca}$ or $\mathbf{c}_a\mathbf{c}_1$	-	$\perp AC$	-
4.	$\mathbf{f}_{bq}$ or $\mathbf{q}_1\mathbf{b}_1$	$\frac{(\mathbf{qb})^2}{QB} = \frac{(1.0)^2}{0.045} = 22.2$	$\parallel QB$	$\rightarrow Q$
5.	$\mathbf{f}^c_{cb}$ or $\mathbf{b}_1\mathbf{c}_b$	$\frac{(\mathbf{bc})^2}{BC} = \frac{(0.12)^2}{0.185} = 0.078$	$\parallel BC$	$\rightarrow B$
6.	$\mathbf{f}^t_{cb}$ or $\mathbf{c}_b\mathbf{c}_a$	-	$\perp BC$	-

Table 10

SN	Vector	Magnitude	Direction	Sense
1.	$\mathbf{f}_{co}$ or $\mathbf{o}_1\,\mathbf{c}_1$	Already drawn	-	-
2.	$\mathbf{f}^c_{dc}$ or $\mathbf{c}_1\,\mathbf{d}_c$	$\frac{(\mathbf{cd})^2}{CD} = \frac{(1.0)^2}{0.24} = 4.17$	$\parallel CD$	$\rightarrow C$
3.	$\mathbf{f}^t_{dc}$ or $\mathbf{d}_c\,\mathbf{d}_1$	-	$\perp CD$	-
4.	$\mathbf{f}_{dg}$ or $\mathbf{g}_1\,\mathbf{d}_1$	-	$\parallel$ to motion of D	-

3.5 CORIOLIS ACCELERATION COMPONENT

It is seen that the acceleration of a moving point relative to a fixed body (fixed coordinate system) may have two components of acceleration; the centripetal and the tangential. However, in some cases, the point may have its motion relative to a moving body (moving coordinate) system, for example, motion of a slider on a rotating link. The following analysis is made to investigate the acceleration at that point.

Let a link AR rotate about a fixed point A on it (Fig. 3.14). P is a point on a slider on the link.

At any given instant,

Let ω = angular velocity of the link

α = angular acceleration of the link

v = linear velocity of the slider on the link

f = linear acceleration of the slider on the link

r = radial distance of point P on the slider

In a short interval of time δt, let $\delta\theta$ be the angular displacement of the link and δr, the radial displacement of the slider in the outward direction.

After the short interval of time δt, let

$\omega' = \omega + \alpha\delta t$ = angular velocity of the link
$v' = v + f.\delta t$ = linear velocity of the slider on the link
$r' = r + \delta r$ = radial distance of the slider

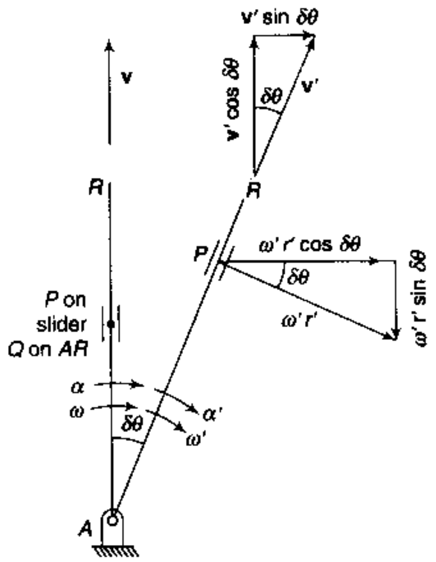


Fig. 3.14

Acceleration of *P* Parallel to *AR*

Initial velocity of P along $AR = v = v_{pq}$
Final velocity of P along $AR = v'\cos\delta\theta - \omega' r' \sin\delta\theta$
Change of velocity along $AR = (v'\cos\delta\theta - \omega' r' \sin\delta\theta) - v$
Acceleration of P along AR

$$= \frac{(v + f\delta t)\cos\delta\theta - (\omega + \alpha\delta t)(r + \delta r)\sin\delta\theta - v}{\delta t}$$

In the limit, as $\delta t \to 0$
$\cos\delta\theta \to 1$ and $\sin\delta\theta \to \delta\theta$

$$\text{Acceleration of } P \text{ along } AR = f - \omega r \frac{d\theta}{dt}$$
$$= f - \omega r w = f - \omega^2 r$$
= Acc. of slider–centripetal. acc.

This is the acceleration of P along AR in the radially outward direction. f will be negative if the slider has deceleration while moving in the outward direction or has acceleration while moving in the inward direction.

Acceleration of P Perpendicular to *AR*

Initial velocity of $P \perp$ to $AR = \omega r$
Final velocity of $P \perp$ to $AR = v' \sin\delta\theta + \omega' r' \cos\delta\theta$
Change of velocity $\perp$ to $AR = (v'\sin\delta\theta + \omega' r'\cos\delta\theta) - \omega r$
Acceleration of $P \perp$ to AR

$$= \frac{(v + f\delta t)\cos\delta\theta - (\omega + \alpha\delta t)(r + \delta r)\cos\delta\theta - \omega r}{}$$

In the limit, as $\delta t \to 0$
$\cos\delta\theta \to 1$ and $\sin\delta\theta \to \delta\theta$

$$\text{Acceleration of } P \perp \text{ to } AR = v\frac{d\theta}{dt} + \omega\frac{dr}{dt} + r\alpha$$
$$= v\omega + \omega v + r\alpha = 2\omega v + r\alpha$$
$$= 2\omega v + \text{tangential acc.}$$

This is the acceleration of P perpendicular to AR. The component $2\omega v$ is known as the *Coriolis acceleration component*. It is positive if both ω and v are either positive or negative.

Thus, the coriolis component is positive if the

- link AR rotates clockwise and the slider moves radially outwards
- link rotates counter-clockwise and the slider moves radially inwards.

Otherwise, the Coriolis component will be negative.

These observations can be summarised into the following rule:

The direction of the Coriolis acceleration component is obtained by rotating the radial velocity vector v through 90° in the direction of rotation of the link.

Let Q be a point on the link AR immediately beneath the point P at the instant. Then acc. of P = acceleration of $P \parallel$ to AR + acceleration of $P \perp$ to AR

$$f_{pa} = (f - \omega^2 r) + (2\omega v + r\alpha)$$

$$= f + (r\alpha - \omega^2 r) + 2\omega v$$

= acc. of P rel. to Q + acc. of Q rel. to A + Coriolis acceleration component

$$= f'_{pq} + f_{qa} + f^{cr} \tag{3.4}$$

In the above equation, f'_{pq} is the acceleration which an observer stationed on link AR would observe for the slider.

In Fig. 3.5(a), the acceleration of the point G relative to the link AD (the acceleration to be reported by a person stationed on the link AD) does not involve the Coriolis component though the link CD has angular motion since G is a fixed point on the link CD. Now in case it is desired to have the acceleration of G relative to the link BC (the acceleration to be reported by a person stationed on the link BC), the Coriolis component of acceleration is involved because now relative to the link BC, G is a moving point and the link BC also has angular velocity. (See Example 3.16).

Remember that Coriolis component exists only if there are two coincident points which have

- linear relative velocity of sliding, and
- angular motion about fixed finite centres of rotation.

Sometimes for the sake of simplicity, it is convenient to associate the Coriolis acceleration component f^{cr} with f'_{pq} and writing the equation in the form,

$$f_{pa} = f_{pq} + f_{qa}$$

where

$$f_{pa} = f'_{pq} + f^{cr} \tag{3.5}$$

This makes solving problems quite easy.

3.6 CRANK AND SLOTTED-LEVER MECHANISM

The configuration and the velocity diagrams of a slotted-lever mechanism have been shown in Figs 3.15(a) and (b) respectively. The crank OP rotates at uniform angular velocity of ω rad/s clockwise.

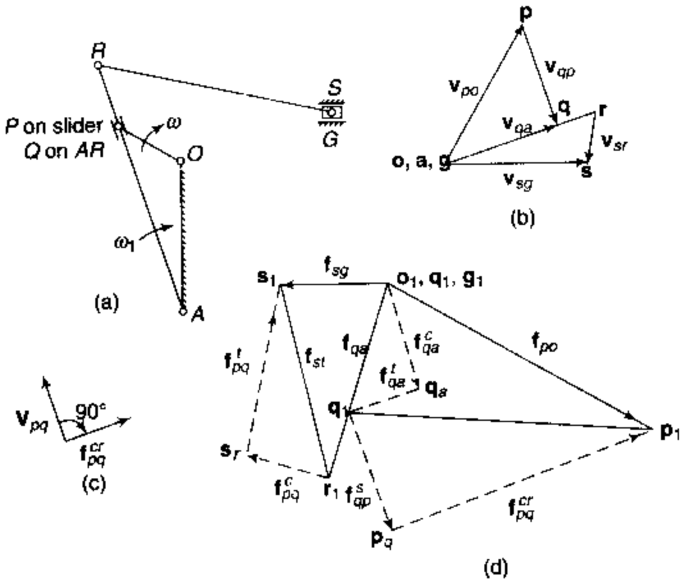


Fig. 3.15

Writing the vector equation.

$$\mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa} \quad \text{or} \quad \mathbf{f}_{qo} = \mathbf{f}_{qp} + \mathbf{f}_{po}$$

Any of these equations can be solved graphically. Both will lead to the same acceleration diagram except for the direction of the vectors $\mathbf{f}_{pq}$ and $\mathbf{f}_{qp}$.

Considering the first equation,

$$\mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa} \quad \text{or} \quad \mathbf{f}_{po} = \mathbf{f}_{qa} + \mathbf{f}_{pq}$$

$$= f^{c}_{qa} + f^{t}_{qa} + f^{s}_{pq} + f^{cr}_{pq}$$

or $\mathbf{o}_1\,\mathbf{p}_1 = \mathbf{a}_1\,\mathbf{q}_a + \mathbf{q}_a\,\mathbf{q}_1 + \mathbf{q}_1\,\mathbf{p}_q + \mathbf{p}_q\,\mathbf{p}_1$

Set the following vector table (Table 11):

The direction is obtained by rotating the vector $\mathbf{v}_{pq}$ (or qp) through 90° in the direction of ω_1 (clockwise in the present case).

Construct the acceleration diagram as follows [Fig. 3.15(c)]:

1. Take the first vector f_{po} which is completely known.
2. Take the second vector from the point a_1 (or o_1). This vector is also completely known.
3. Only the direction of the third vector f^{t}_{qa} is known. Draw a line ⊥ to AQ through the head $\mathbf{q}_a$ of the second vector.
4. As the head of the third vector is not available, the fourth vector cannot be added to it.

   Take the last vector f^{cr}_{pq} which is completely known. Place this vector in the proper direction and sense so that $\mathbf{p}_1$ becomes the head of the vector. In Fig. 3.15(d), $\mathbf{p}_q$ cannot lie on the right side of $\mathbf{p}_3$ because then the vector would become $\mathbf{p}_1\,\mathbf{p}_q$ and not $\mathbf{p}_q\,\mathbf{p}_1$.
5. For the fourth vector, draw a line parallel to AR through the point $\mathbf{p}_q$ of the fifth vector.

   The intersection of this line with the line drawn in the step 3 locates the point $\mathbf{q}_1$.

   Total acc. of P rel. to Q, $\mathbf{f}_{pq} = \mathbf{q}_1\,\mathbf{p}_1$

   Total acc. of Q rel. to A, $\mathbf{f}_{qa} = \mathbf{a}_1\,\mathbf{q}_1$

   The acceleration of R relative to A is given on $\mathbf{a}_1\,\mathbf{q}_1$ produced such that

$$\frac{\mathbf{a}_1\mathbf{r}_1}{\mathbf{a}_1\mathbf{q}_1} = \frac{AR}{AQ}$$

Table 11

SN	Vector	Magnitude	Direction	Sense
1.	$\mathbf{f}_{po}$ or $\mathbf{o}_1\,\mathbf{p}_1$	$\omega \times OP$	$\parallel OP$	$\to O$
2.	$\mathbf{f}^{c}_{qa}$ or $\mathbf{a}_1\,\mathbf{q}_a$	$\frac{(\mathbf{aq})^2}{AQ}$	$\parallel AQ$	$\to A$
3.	$\mathbf{f}^{t}_{qa}$ or $\mathbf{q}_a\,\mathbf{q}_1$	-	$\perp AQ$ or $\mathbf{a}_1\,\mathbf{q}_a$	-
4.	$\mathbf{f}^{s}_{pq}$ or $\mathbf{q}_1\,\mathbf{p}_q$	-	$\parallel AR$	-
5.	$\mathbf{f}^{cr}_{pq}$ or $\mathbf{p}_q\,\mathbf{p}_1$	Coriolis component*	$\perp AR$	Refer*

* $f^{cr}_{pq} = 2\omega_1 v_{pq}$ (ω_1 = angular vel. of AR) $= 2\left(\frac{\mathbf{aq}}{AQ}\right)qp$

Note that in the present case, the sliding acceleration $\mathbf{a_1q_a}$ is in the opposite direction to the sliding velocity **qp**. This signifies that the slider is decelerating.

Also,

$$\mathbf{f}_{sa} = \mathbf{f}_{sr} + \mathbf{f}_{ra}$$
$$\mathbf{f}_{sg} = \mathbf{f}_{ra} + \mathbf{f}_{sr}$$
$$= \mathbf{f}_{ra} + \mathbf{f}^{c}_{sr} + \mathbf{f}^{t}_{sr}$$
$$\mathbf{g_1\,s_1} = \mathbf{a_1\,r_1} + \mathbf{r_1\,s_r} + \mathbf{s_r\,s_1}$$

This equation can be solved as usual.

Total acc. of S relative to R, $\mathbf{f}_{sr} = \mathbf{r_1\,s_1}$

Acceleration of $S = \mathbf{g_1\,s_1}$ or $\mathbf{a_1\,s_1}$ or $\mathbf{o_1\,s_1}$

The direction of $\mathbf{g_1s_1}$ is opposite to the direction of motion of the slider S indicating that the slider is decelerating.

Example 3.9 *Figure 3.16(a) shows a slider moving outwards on a rod with a velocity of 4 m/s when its distance from the point O is 1.5 m. At this instant, the velocity of the slider is increasing at a rate of 10 m/s². The rod has an angular velocity of 6 rad/s counter-clockwise about O and an angular acceleration of 20 rad/s² clockwise. Determine the absolute acceleration of the slider.*

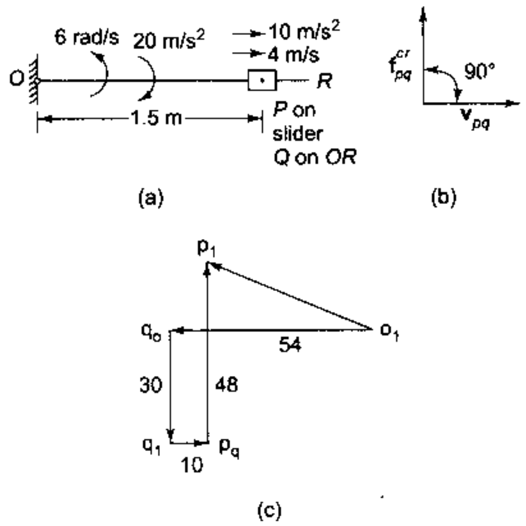


Fig. 3.16

Solution

Writing the acceleration vector equation,

$$\mathbf{f}_{po} = \mathbf{f}_{pq} + \mathbf{f}_{qo} = \mathbf{f}_{qo} + \mathbf{f}_{pq} = \mathbf{f}^{c}_{qo} + \mathbf{f}^{t}_{qo} + \mathbf{f}^{s}_{pq} + \mathbf{f}^{cr}_{pq}$$

or $\quad \mathbf{o_1\,p_1} = \mathbf{o_1q_o} + \mathbf{q_oq_1} + \mathbf{q_1p_q} + \mathbf{p_q\,p_1}$

Set the following vector table (Table 12):

Figure 3.16(b) shows how to obtain the direction of the coriolis component. The velocity vector of the slider is rotated through 90° in the angular direction of the rod.

Draw the acceleration diagram as follows [Fig. 3.16(c)]:

1. From the pole point $\mathbf{o_1}$, take the first vector $\mathbf{o_1q_o}$.
2. Add to it the second vector $\mathbf{q_oq_1}$.

Table 12

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^{c}_{qo}$ or $\mathbf{o_1\,q_o}$	$\omega^2.r = 6^2 \times 1.5 = 54$	$\parallel OR$	←
2.	$\mathbf{f}^{t}_{qo}$ or $\mathbf{q_o\,q_1}$	$\alpha_{or} \times OQ = 20 \times 1.5 = 30$	$\perp OR$	↓
3.	$\mathbf{f}^{s}_{pq}$ or $\mathbf{q_1\,p_q}$	10	$\parallel OR$	→
4.	$\mathbf{f}^{cr}_{pq}$ or $\mathbf{p_q\,p_1}$	$2\omega.\,v_{pq} = 2 \times 6 \times 4 = 48$	$\perp OR$	

3. Add the third vector to the second vector.
4. To add the fourth vector, place the tail of the fourth vector at the head of the third vector and the final point $\mathbf{p}_1$ is located.
5. Join $\mathbf{o}_1\,\mathbf{p}_1$.
   On measurement, $\mathbf{o}_1\,\mathbf{p}_1 = 47.5$ m/s^2
   or by calculation from the acceleration diagram
   $= \sqrt{(54-10)^2+(48-30)^2} = 47.5$ m/s^2

Example 3.10 *Fig. 3.17(a) shows the link mechanism of a quick-return mechanism of the slotted-lever type, the various dimensions of which are*

OA = 400 mm, OP = 200 mm, AR = 700 mm, RS = 300 mm.

For the configuration shown, determine the acceleration of the cutting tool at S and the angular acceleration of the link RS. The crank OP rotates at 210 rpm.

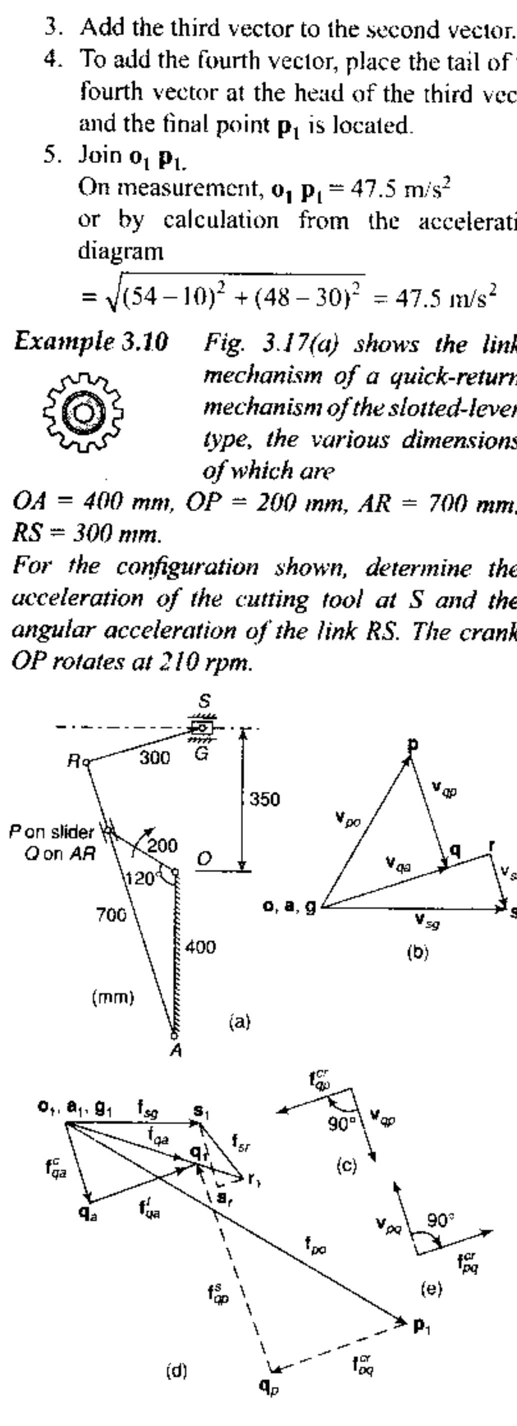


Fig. 3.17

Solution The velocity diagram has been reproduced in Fig. 3.17(b) from Fig. 2.21.

$$v_{po} \text{ or } \mathbf{op} = \omega.OP = \frac{2\pi\times 210}{60}\times 0.2 = 22\times 0.2 = 4.4 \text{ m/s}$$

Writing the acceleration vector equation,

$$\mathbf{f}_{qo} = \mathbf{f}_{qp} + \mathbf{f}_{po}$$
$$\mathbf{f}_{qa} = \mathbf{f}_{po} + \mathbf{f}_{qp}$$

or $\quad \mathbf{a}_1\,\mathbf{q}_1 = \mathbf{o}_1\,\mathbf{p}_1 + \mathbf{p}_1\,\mathbf{q}_1$

or $\quad \mathbf{f}_{qa}^c + \mathbf{f}_{qa}^t = \mathbf{f}_{po}^c + \mathbf{f}_{qp}^{cr} + \mathbf{f}_{qp}^s$

or $\quad \mathbf{a}_1\,\mathbf{q}_a + \mathbf{q}_a\,\mathbf{q}_1 = \mathbf{o}_1\,\mathbf{p}_1 + \mathbf{p}_1\,\mathbf{q}_p + \mathbf{q}_p\,\mathbf{q}_1$

Set the following vector table (Table 13) shown in the following page.

Direction of f_{qp}^{cr} is obtained by rotating $\mathbf{v}_{qp}$ through 90° in the direction of angular movement of link QA as shown in Fig. 3.17(c) (clockwise in this case). Draw the acceleration diagram as follows [Fig. 3.17(d)]:

1. From the pole point $\mathbf{o}_1$, take the first vector $\mathbf{o}_1\,\mathbf{p}_1$.
2. Add to it the second vector $\mathbf{p}_1\mathbf{q}_p$.
3. Add the third vector to the second vector. For the fourth vector, draw a line parallel to AQ through the head $\mathbf{q}_p$ of the third vector.
4. From the pole point $\mathbf{a}_1$ or $\mathbf{o}_1$, take the fifth vector and for the sixth vector, draw a line perpendicular to AQ through the head $\mathbf{q}_a$ of the fifth vector.
   This way the point $\mathbf{q}_1$ is located.
5. Join $\mathbf{q}_1$ and $\mathbf{a}_1$ and extend to $\mathbf{r}_1$ such that
   $$\frac{\mathbf{a}_1\mathbf{r}_1}{\mathbf{a}_1\mathbf{q}_1} = \frac{AR}{AQ}$$
   Writing the vector equation,
   $$\mathbf{f}_{so} = \mathbf{f}_{sr} + \mathbf{f}_{ro}$$
   or $\quad \mathbf{f}_{sg} = \mathbf{f}_{ro} + \mathbf{f}_{sr}$
   $$= \mathbf{f}_{ro} + \mathbf{f}_{sr}^c + \mathbf{f}_{sr}^t$$
   or $\quad \mathbf{g}_1\,\mathbf{s}_1 = \mathbf{o}_1\,\mathbf{r}_1 + \mathbf{r}_1\,\mathbf{s}_r + \mathbf{s}_r\,\mathbf{s}_1$

$\mathbf{f}_{ro}$ is already available on the acceleration diagram. $\mathbf{f}_{sg}$ is horizontal.

$$\mathbf{f}_{sr}^c = \frac{(\mathbf{rs})^2}{RS} = \frac{(1.41)^2}{0.3} = 6.63 \text{ m/s}^2$$

Complete the vector diagram as usual.

Acceleration of the cutting tool, $\mathbf{f}_s = \mathbf{o}_1\,\mathbf{s}_1$ = <u>32.8 m/s^2</u>

$$\alpha_{rs} = \frac{\mathbf{f}_{rs}^t \text{ or } \mathbf{s}_1\mathbf{s}_r}{RS} = \frac{15.7}{0.3} = \underline{52.3 \text{ rad/s}} \text{ clockwise}$$

Table 13

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^c_{po}$ or $\mathbf{o_1 p_1}$	$\frac{(\mathbf{op})^2}{OP} = \frac{(4.4)^2}{0.2} = 96.8$	$\parallel OP$	$\rightarrow O$
2.	$\mathbf{f}^{cr}_{qp}$ or $\mathbf{p_1 q_p}$	$2\,\omega_{ra}\,\mathbf{v}_{qp} = 35.5^*$	$\perp AQ$	Refer *
3.	$\mathbf{f}^s_{qp}$ or $\mathbf{q_p q_1}$	–	$\parallel AQ$	–
4.	$\mathbf{f}^c_{qa}$ or $\mathbf{a_1 q_a}$	$\frac{(\mathbf{aq})^2}{AQ} = \frac{(3.26)^2}{0.52} = 20.4$	$\parallel AQ$	$\rightarrow A$
5.	$\mathbf{f}^t_{qa}$ or $\mathbf{q_a q_1}$	–	$\perp AQ$	–

$$^*\mathbf{f}^{cr}_{qp} = 2\omega_{ra}v_{qp} = 2\frac{v_{ra}}{RA}v_{qp} = 2 \times \frac{4.36}{0.7} \times 2.85 = 35.5 \text{ m/s}^2$$

Note: In case the problem is to be worked out without writing the vector equation and if the Coriolis acceleration component $\mathbf{f}^{cr}_{pq}$ is considered instead of $\mathbf{f}^{cr}_{qp}$, then note that

- the magnitude of the Coriolis component remains the same.
- in order to find the direction, the velocity vector $\mathbf{v}_{qp}$ is to be rotated through 90° as shown in Fig. 3.17e. The direction of $\mathbf{f}^{cr}_{qp}$ is found to be opposite to $\mathbf{f}^{cr}_{pq}$. Now, one will be tempted to place this vector towards right of $\mathbf{p}_1$ in the acceleration diagram. However, if that is done, the vector would be read as $\mathbf{p_1 q_p}$ which means $\mathbf{f}^{cr}_{qp}$ and not $\mathbf{f}^{cr}_{pq}$. Thus, again the vector $\mathbf{f}^{cr}_{qp}$ has to be placed at the same place, i.e., on the left of $\mathbf{p}_1$ which means the acceleration diagram obtained will be the same.

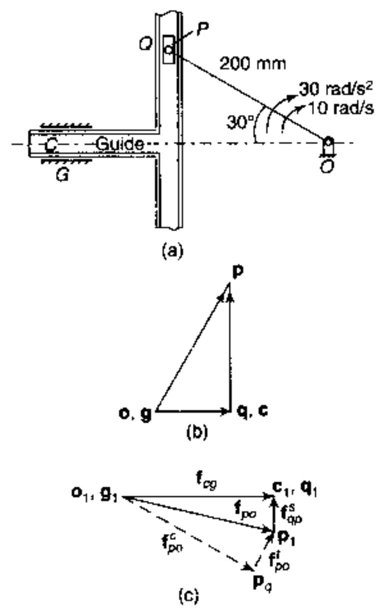


Fig. 3.18

Example 3.11 *Figure 3.18(a) shows the Scotch yoke mechanism. At the instant shown in the figure, the crank OP has an angular velocity of 10 rad/s and an angular acceleration of 30 rad/s². Determine the acceleration of the slider P in the guide and the horizontal acceleration of the guide.*

Solution $v_{po} = 10 \times 0.2 = 2$ m/s

To draw the velocity diagram, take a coincident point Q just beneath P on the guide link. Take another point C on the guide link. Now, proceed as follows [Fig. 3.18(b)]:

1. Take the vector **op** equal to 2 m/s to some suitable scale.
2. The velocity of Q relative to P is along the guide path. Therefore, draw a line parallel to this path (vertical) through **p** to locate the point **q**.
3. The velocity of C relative to G is along the guide path at G or is horizontal. Thus, draw a horizontal line through **g** to locate point **c**.
4. Now, Q and C are two fixed points on the same link and the distance between them does not vary. Therefore, the points **q** and **c** in the velocity diagram coincide. Thus, the intersection of lines drawn in steps 2 and 3 locates points **q** or **c**.

Now, $\mathbf{f}^c_{po}$ or $\mathbf{o}_1\,\mathbf{p}_o = \dfrac{(\mathbf{op})^2}{OP} = \dfrac{2^2}{0.2} = 20\text{ m/s}^2$

$\mathbf{f}^t_{po}$ or $\mathbf{p}_o\,\mathbf{p}_1 = 30 \times 0.2 = 6\text{ m/s}^2$

Draw acceleration diagram as follows [Fig. 3.18(c)]:

1. First take the centripetal acceleration component $\mathbf{f}^c_{po}$ or $\mathbf{o}_1\mathbf{p}_o$ and add the tangential component $\mathbf{f}^t_{po}$ or $\mathbf{p}_o\mathbf{p}_1$ to it.
2. Now, the linear acceleration of sliding of Q relative to P is vertical. Thus, draw a line to locate point $\mathbf{q}_1$ on that.
3. Draw a horizontal line through $\mathbf{g}_1$ to locate the point $\mathbf{c}_1$ on that.
4. As there is zero velocity between Q and P, they are to be the coinciding points in the acceleration diagram also. Thus, the intersection of lines drawn in steps 2 and 3 locates the point $\mathbf{q}_1$ or $\mathbf{c}_1$.

Acceleration of slider P = $\mathbf{f}_{pq}$ or $\mathbf{q}_1\mathbf{p}_1$ $= 4.75\text{ m/s}^2$

and horizontal acceleration of guide $= \mathbf{f}_{cg}$ or $\mathbf{g}_1\,\mathbf{c}_1 = 20.5\text{ m/s}^2$

It is to be noted that in this example, Q and P are two coincident points, but still there is no Coriolis component. This is because the link (guide) on which the slider is moving does not have any angular motion and thus ω for that is zero.

Example 3.12 ***A Whitworth quick-return mechanism has been shown in Fig. 3.19(a). The dimensions of the links are OP (crank) = 240 mm, OA = 150 mm, AR = 165 mm and RS = 430 mm. The crank OP has an angular velocity of 2.5 rad/s and an angular deceleration of 20 rad/s² at the instant. Determine the***

(i) acceleration of the slider S

(ii) angular acceleration of links AR and RS

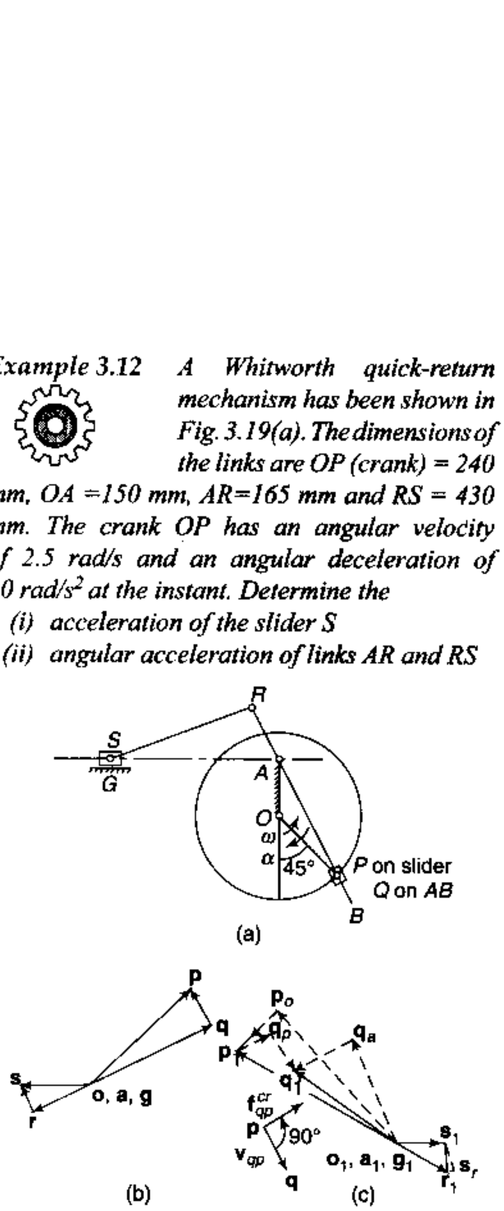


Fig. 3.19

Solution The velocity diagram has been reproduced in Figs 3.19(b) from Fig. 2.25(b).

Writing the acceleration vector equation,

$$\mathbf{f}_{qo} = \mathbf{f}_{qp} + \mathbf{f}_{po} \qquad \text{or} \qquad \mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa}$$

Both the equations lead to the same acceleration diagram except that the direction sense of the acceleration vectors between P and Q are reversed.

Taking the first one,

$$\mathbf{f}_{qo} = \mathbf{f}_{qp} + \mathbf{f}_{po}$$

$$\mathbf{f}_{qa} = \mathbf{f}_{po} + \mathbf{f}_{qp}$$

or $\quad \mathbf{a}_1\,\mathbf{q}_1 = \mathbf{o}_1\,\mathbf{p}_1 + \mathbf{p}_1\,\mathbf{q}_1$

Each has two components,

$$\mathbf{f}^c_{qa} + \mathbf{f}^t_{qa} = \mathbf{f}^c_{po} + \mathbf{f}^t_{po} + \mathbf{f}^{cr}_{qp} + \mathbf{f}^s_{qp}$$

or $\mathbf{a_1 q_a + q_a q_1 = o_1 p_o + p_o p_1 + p_1 q_p + q_p q_1}$

Set the following vector table (Table 14):

The direction of f_{qp}^{cr} is obtained by rotating $\mathbf{v}_{qp}$ through 90° in the direction of angular movement of the link QA or BA (counter-clockwise in this case) Draw the acceleration diagram as follows [Fig. 3.19(c)]:

1. From the pole point $\mathbf{o_1}$, take the first vector and add to it the second vector.
2. Add the third vector to the second vector. For the fourth vector, draw a line parallel to AQ, through the head $\mathbf{q_p}$ of the third vector.
3. From the pole point $\mathbf{a_1}$ or $\mathbf{o_1}$, take the fifth vector and for the sixth vector, draw a line perpendicular to AQ through the head $\mathbf{q_a}$ of the fifth vector.
   This way the point $\mathbf{q_1}$ is located.
4. Join $\mathbf{q_1}$ and $\mathbf{a_1}$ and extend to $\mathbf{r_1}$ such that
   $$\frac{\mathbf{a_1 r_1}}{\mathbf{a_1 q_1}} = \frac{AR}{AQ}$$
   Writing the vector equation,
   $$\mathbf{f}_{so} = \mathbf{f}_{sr} + \mathbf{f}_{ro}$$
   or $$\mathbf{f}_{sg} = \mathbf{f}_{ro} + \mathbf{f}_{sr}$$
   $$= \mathbf{f}_{ro} + \mathbf{f}_{sr}^{c} + \mathbf{f}_{sr}^{t}$$
   or $$\mathbf{g_1 s_1 = o_1 r_1 + r_1 s_r + s_r s_1}$$
   $\mathbf{f}_{ro}$ is already available on the acceleration diagram. $\mathbf{f}_{sg}$ is horizontal.

$$\mathbf{f}_{sr}^{c} = \frac{(\mathbf{rs})^2}{RS} = \frac{(0.12)^2}{0.43} = 0.033 \text{ m/s}^2$$

Complete the vector diagram as usual.

$$\mathbf{f_s = o_1 s_1} = 0.39 \text{ m/s}^2$$

$$\alpha_{ar} = \alpha_{qa} = \frac{\mathbf{f}_{qa}^{t} \text{ or } \mathbf{q}_a\, \mathbf{q}_1}{QA} = \frac{0.57}{0.365}$$

$= \underline{1.56 \text{ rad/s}^2}$ clockwise

$$\alpha_{rs} = \frac{\mathbf{f}_{rs}^{t} \text{ or } \mathbf{s}_1\, \mathbf{s}_r}{RS} = \frac{0.24}{0.43}$$

$= \underline{0.558 \text{ rad/s}^2}$ clockwise

Example 3.13 *One cylinder of a rotary engine is shown in the configuration diagram shown in Fig. 3.20(a). OA is the fixed crank, 200 mm long. OP is the connecting rod and is 520 mm long. The line of stroke is along AR and at the instant is inclined at 30° to the vertical. The body of the engine consisting of cylinders rotates at a uniform speed of 400 rpm about the fixed centre A. Determine the*

(i) acceleration of piston (slider) inside the cylinder

(ii) angular acceleration of the connecting rod

Table 14

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}_{po}^{c}$ or $\mathbf{o_1 p_o}$	$\frac{(\mathbf{op})^2}{OP} = \frac{(0.6)^2}{0.24} = 1.5$	$\parallel OP$	$\rightarrow O$
2.	$\mathbf{f}_{po}^{t}$ or $\mathbf{p_o p_1}$	$\alpha_{op} \times OP = 20 \times 0.24 = 0.48$	$\perp OP$ or $\parallel \mathbf{op}$	$\rightarrow \mathbf{o}$
3.	$\mathbf{f}_{qp}^{cr}$ or $\mathbf{p_1 q_p}$	$2\,\omega_{ba}\,\mathbf{v}_{qp} = 0.38$*	$\perp AQ$	Refer *
4.	$\mathbf{f}_{qp}^{s}$ or $\mathbf{q_p q_1}$	-	$\parallel AQ$	-
5.	$\mathbf{f}_{qa}^{c}$ or $\mathbf{a_1 q_a}$	$\frac{(\mathbf{aq})^2}{\mathbf{AQ}} = \frac{(0.585)^2}{0.365} = 0.93$	$\parallel AQ$	$\rightarrow A$
6.	$\mathbf{f}_{qa}^{t}$ or $\mathbf{q_a q_1}$	-	$\perp AQ$	-

$$^*\mathbf{f}_{qp}^{cr} = 2\omega_{ba} v_{qp} = 2\frac{v_{qa}}{\mathbf{QA}} v_{qp} \quad (\omega_{ba} = \omega_{qa}) = 2 \times \frac{0.585}{0.365} \times 0.118 \;\; (v_{qa} = \mathbf{aq} = 0.585) = 0.38 \text{ m/s}^2$$

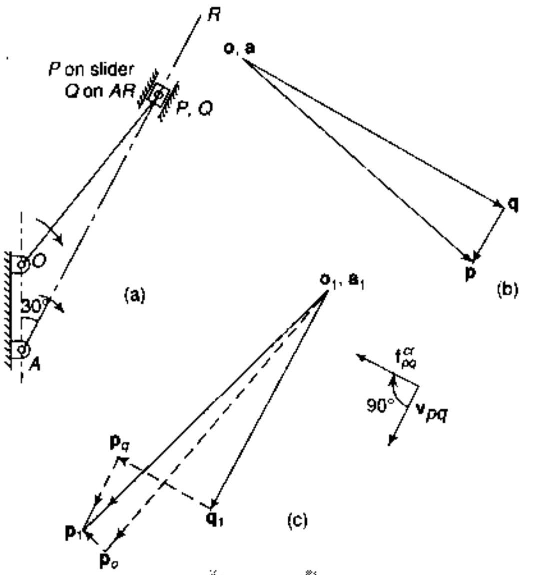


Fig. 3.20

Solution Let Q be a point on AR beneath the point P.

$$v_{qa} = \frac{2\pi N}{60} \times QA = \frac{2\pi \times 400}{60} \times 0.68 = 28.5 \text{ m/s}$$

The velocity vector equation is

$$\mathbf{v}_{pa} = \mathbf{v}_{pq} + \mathbf{v}_{qa} \quad \text{or} \quad \mathbf{v}_{qo} = \mathbf{v}_{qp} + \mathbf{v}_{po}$$

Taking the first one,

$$\mathbf{v}_{po} = \mathbf{v}_{qa} + \mathbf{v}_{qp}$$

or $\quad \mathbf{op} = \mathbf{aq} + \mathbf{qp}$

Take the vector $\mathbf{v}_{qa}$ to a convenient scale [Fig. 3.20(b)].

$\mathbf{v}_{pq}$ is || to AR, draw a line || to AR through **q**.

$\mathbf{v}_{po}$ is ⊥ to OP, draw **op**, a line ⊥ to OP through **o**.

The intersection locates the point **p**.

Similarly, writing the acceleration vector equation,

$$\mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa}$$

or $\quad \mathbf{f}_{po} = \mathbf{f}_{qa} + \mathbf{f}_{pq}$

or $\quad \mathbf{o_1 p_1} = \mathbf{a_1 q_1} + \mathbf{q_1 p_1}$

Expanding, $\mathbf{f}^c_{po} + \mathbf{f}^t_{po} = \mathbf{f}_{qa} + \mathbf{f}^{cr}_{pq} + \mathbf{f}^s_{pq}$

or $\quad \mathbf{o_1 p_o} + \mathbf{p_o p_1} = \mathbf{a_1 q_1} + \mathbf{q_1 p_q} + \mathbf{p_q} + \mathbf{p_1}$

Set the vector table (Table 15):

The direction of $\mathbf{f}^{cr}_{pq}$ is obtained by rotating $\mathbf{v}_{pq}$ through 90° in the direction of ω_{qa} (clockwise). Draw the vector diagram as follows:

1. Take the first vector from the pole point $\mathbf{a}_1$ or $\mathbf{o}_1$ [Fig. 3.20(c)].
2. Add the second vector to the first vector.
3. Through the head of the second vector, draw a line parallel to AQ for the third vector.
4. Take the fourth vector from the pole point $\mathbf{o}_1$.
5. Through the head of the fourth vector, draw a line perpendicular to OP for the fifth vector. The intersection of the lines drawn in steps (3) and (5) locates the point $\mathbf{p}_1$.

(i) Acceleration of the slider inside the cylinder

$\mathbf{f}^s_{pq}$ or $\mathbf{p_q\ p_1}$ = <u>390 m/s²</u>

(ii) Angular acceleration of the connecting rod

$$\alpha_{op} = \frac{\mathbf{f}^t_{po} \text{ or } \mathbf{p_o p_1}}{OP} = \frac{150}{0.52} = \underline{288.5 \text{ rad/s}^2}$$

counter-clockwise

Table 15

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}_{qa}$ or $\mathbf{a_1q_1}$	$\frac{(\mathbf{aq})^2}{AQ} = \frac{(28.5)^2}{0.68} = 1194$	$\parallel AQ$	$\to A$
2.	$\mathbf{f}^{cr}_{pq}$ or $\mathbf{q_1\ p_q}$	486*	$\perp AQ$	-
3.	$\mathbf{f}^s_{pq}$ or $\mathbf{p_q\ p_1}$	-	$\parallel AQ$	-
4.	$\mathbf{f}^c_{po}$ or $\mathbf{o_1\ p_o}$	$\frac{(\mathbf{op})^2}{OP} = \frac{(29.3)^2}{0.52} = 1651$	$\parallel OP$	$\to \mathbf{o}$
5.	$\mathbf{f}^t_{po}$ or $\mathbf{p_o\ p_1}$	-	$\perp OP$	-

$$^*\mathbf{f}^{cr}_{pq} = 2\omega_{ra}v_{pq} = 2\frac{v_{qa}}{QA}\mathbf{qp} = 2 \times \frac{28.5}{0.68} \times 5.8 = 486 \text{ m/s}^2$$

Example 3.14 *In the swiveling-joint mechanism shown in Fig. 3.21(a), AB is the driving crank rotating at 300 rpm clockwise. The lengths of the various links are*

AD = 650 mm, AB = 100 mm, BC = 800 mm, DC = 250 mm, BE = CE, EF = 400 mm, OF = 240 mm, FS = 400 mm

For the given configuration of the mechanism, determine the acceleration of sliding of the link EF in the trunnion.

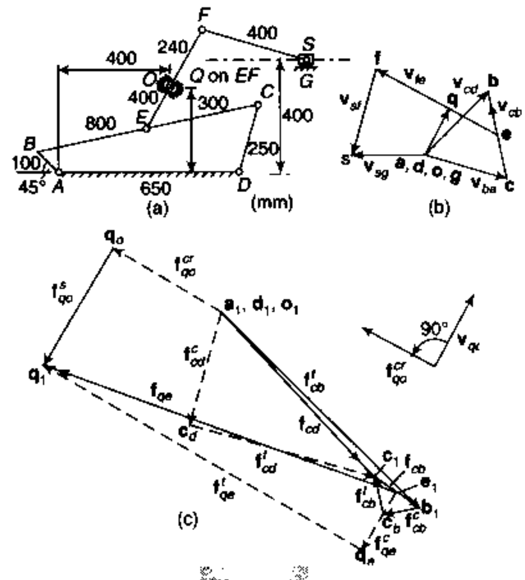


Fig. 3.21

Solution

$$\omega_{po} = \frac{2\pi \times 300}{60} = 31.4 \text{ rad/s}$$

$$v_b = 31.4 \times 0.1 = 3.14 \text{ m/s}$$

The velocity diagram is reproduced in Fig. 3.21(b) from Example 2.18. For the acceleration diagram, first complete the acceleration diagram for the four-link mechanism *ABCD* [Fig. 3.21(c)] as usual with the help of the following equation and table:

$$\mathbf{f}_{ca} = \mathbf{f}_{cb} + \mathbf{f}_{ba}$$

$$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

$$\mathbf{f}^c_{cd} + \mathbf{f}^t_{cd} = \mathbf{f}^c_{ba} + \mathbf{f}^c_{cb} + \mathbf{f}^t_{cb}$$

or $\quad \mathbf{d}_1\mathbf{c}_d + \mathbf{c}_d\mathbf{c}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{c}_b + \mathbf{c}_b\mathbf{c}_1$

Set the vector table (Table 16) as shown.

Now, locate the point $\mathbf{e}_1$, on the vector $\mathbf{b}_1\mathbf{c}_1$ at the mid point.

Acceleration of *Q* relative to *E* has two components:

(i) $\mathbf{f}^c_{qe} = \dfrac{v_{qe}}{QE} = \dfrac{(1.95)^2}{0.16} = 23.8 \text{ m/s}^2, \parallel QE$

(ii) $\mathbf{f}^t_{qe}$ is unknown in magnitude, and its direction is $\perp QE$.

From the point $\mathbf{e}_1$, take the vector for $\mathbf{f}^c_{qe}$ parallel to *QE* and draw a line ⊥ to it for the vector $\mathbf{f}^d_{qe}$.

Now, the Coriolis component can be calculated,

$$* \, \mathbf{f}^{cr}_{qe} = 2\omega_{qe}v_{qo} = 2\frac{v_{qe}}{QE}v_{qo}$$

Table 16

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^c_{ba}$ or $\mathbf{a}_1\mathbf{b}_1$	$\dfrac{(\mathbf{ab})^2}{AB} = \dfrac{(3.14)^2}{0.1} = 98.6$	∥ *AB*	→ *A*
2.	$\mathbf{f}^c_{cb}$ or $\mathbf{b}_1\mathbf{c}_b$	$\dfrac{(\mathbf{bc})^2}{BC} = \dfrac{(3.17)^2}{0.8} = 12.6$	∥ *BC*	→ *B*
3.	$\mathbf{f}^t_{cb}$ or $\mathbf{c}_b\mathbf{c}_1$	-	⊥ *BC*	-
4.	$\mathbf{f}^c_{cd}$ or $\mathbf{d}_1\mathbf{c}_d$	$\dfrac{(\mathbf{dc})^2}{DC} = \dfrac{(3.18)^2}{0.25} = 40.5$	∥ *DC*	→ *D*
5.	$\mathbf{f}^t_{cd}$ or $\mathbf{c}_d\mathbf{c}_1$	-	⊥ *DC*	-

$$= 2 \times \frac{1.95}{0.16} \times 1.85 = 45.1 \text{ m/s}^2$$

ω_{qe} is found to be counter-clockwise.

The direction for Coriolis component is taken by rotating v_{qe} through 90° in the direction of angular movement of the link *QE* (counter-clockwise in this case). The acceleration diagram is completed as usual.

Acceleration of sliding of link *EF* in the trunnion = $\mathbf{q_o q_1}$ = 4.86 m/s²

This shows that it is downwards or opposite to the velocity. Thus, it is deceleration.

Example 3.15 *In the pump mechanism shown in Fig. 3.22(a), OA = 320 mm, AC = 680 mm and OQ = 650 mm. For the given configuration, determine*

(i) linear (sliding) acceleration of slider C relative to cylinder walls

(ii) angular acceleration of the piston rod

Solution The velocity diagram has been reproduced in Fig. 3.22(b) from Example 2.15.

The problem can be solved by either of the two methods discussed for velocity diagram in Example 2.15.

Writing the acceleration vector equation for the latter configuration,

$$\mathbf{f}_{aq} = \mathbf{f}_{ab} + \mathbf{f}_{bq}$$

or
$$\mathbf{f}_{ao} = \mathbf{f}_{bq} + \mathbf{f}_{ab}$$

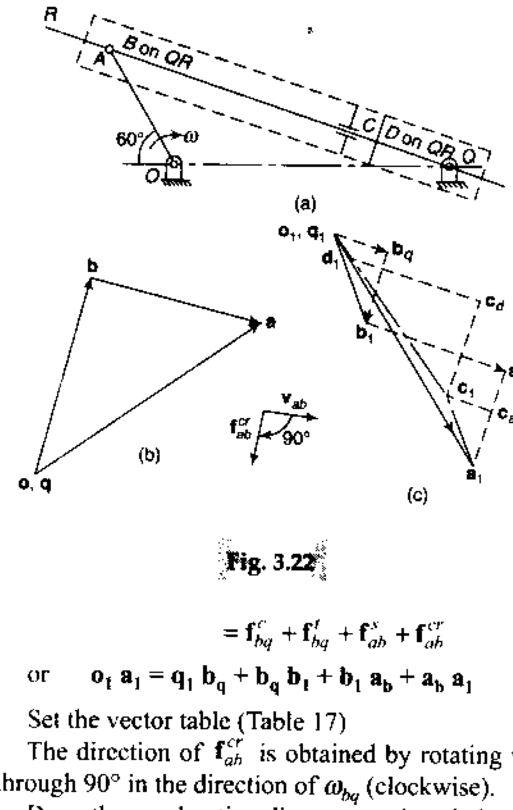


Fig. 3.22

$$= \mathbf{f}^c_{bq} + \mathbf{f}^t_{bq} + \mathbf{f}^s_{ab} + \mathbf{f}^{cr}_{ab}$$

or
$$\mathbf{o_1 a_1} = \mathbf{q_1 b_q} + \mathbf{b_q b_1} + \mathbf{b_1 a_b} + \mathbf{a_b a_1}$$

Set the vector table (Table 17)

The direction of $\mathbf{f}^{cr}_{ab}$ is obtained by rotating $\mathbf{v}_{ba}$ through 90° in the direction of ω_{bq} (clockwise).

Draw the acceleration diagram as given below:

1. Take the first vector [Fig. 3.22(c)].
2. From point $\mathbf{q}_1$ (pole point), take the second vector and through the head of it, draw a line perpendicular to *QB* for the third vector.

Table 17

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f_{ao}}$ or $\mathbf{o_1 a_1}$	$\frac{(\mathbf{oa})^2}{OA} = \frac{(6.4)^2}{0.32} = 128$	∥ *OA*	→ O
2.	$\mathbf{f^c_{bq}}$ or $\mathbf{q_1 b_q}$	$\frac{(\mathbf{bq})^2}{BQ} = \frac{(4.77)^2}{0.85} = 26.8$	∥ *QB*	→ Q
3.	$\mathbf{f^t_{bq}}$ or $\mathbf{b_q b_1}$	-	⊥ *QB*	-
4.	$\mathbf{f^s_{ab}}$ or $\mathbf{b_1 f_b}$	-	∥ *QB*	-
5.	$\mathbf{f^{cr}_{ab}}$ or $\mathbf{a_b a_1}$	47.1*	⊥ *QB*	-

$$^*\mathbf{f}^{cr}_{ab} = 2\omega_{rq} v_{ab} = 2\frac{v_{bq}}{BQ}\mathbf{ba} \quad (\omega_{rq} = \omega_{bq}) = 2 \times \frac{4.77}{0.85} \times 4.2 = 47.1 \text{ m/s}^2$$

3. Take the fifth vector from the point $\mathbf{a}_1$ such that the vector is in the proper direction and sense.
4. For the fourth vector, draw a line parallel to QB through the tail $\mathbf{a}_b$ of the fifth vector.

The intersection of the lines drawn in steps (2) and (4) locates point $\mathbf{b}_1$.

Let D be a point on QB beneath the point C.

Acc. of C rel. to cylinder walls

$= f^s_{cd} = f^s_{ab} = \underline{72 \text{ m/s}^2}$

Writing the vector equation,

$$\mathbf{f}_{cq} = \mathbf{f}_{cd} + \mathbf{f}_{dq}$$
$$= \mathbf{f}_{dq} + \mathbf{f}_{cd}$$
$$= \mathbf{f}_{dq} + \mathbf{f}^s_{cd} + \mathbf{f}^{cr}_{cd}$$

or $$\mathbf{q}_1\mathbf{c}_1 = \mathbf{q}_1\mathbf{d}_1 + \mathbf{d}_1\mathbf{c}_d + \mathbf{c}_d\mathbf{d}_1$$

$\mathbf{f}^s_{cd} = \mathbf{f}^s_{ab} = \underline{72 \text{ m/s}^2}$ parallel to QB.

$\mathbf{f}^{cr}_{cd} = 2\omega_{rq} v_{cd} = 2\omega_{rq} v_{ab} = f^{cr}_{ab}$

Locate point $\mathbf{d}_1$ on $\mathbf{q}_1\mathbf{b}_1$ such that $\dfrac{\mathbf{q}_1\mathbf{d}_1}{\mathbf{q}_1\mathbf{b}_1} = \dfrac{QD}{QB}$

To the vector $\mathbf{f}_{dq}$, add the vector $\mathbf{f}^s_{cd}$ and then $\mathbf{f}^{cr}_{cd}$.

Thus, the point $\mathbf{c}_1$ is located.

Join $\mathbf{a}_1$ to $\mathbf{c}_1$.

$\mathbf{a}_1\mathbf{c}_1$ represents total acceleration of C relative to A. This has two components.

$\mathbf{f}^t_{cd} \perp$ to $CA = \mathbf{a}_1\mathbf{c}_a$

and $\mathbf{f}^c_{ca} \parallel$ to CA ($= \mathbf{c}_a\mathbf{c}_1$)

$$\alpha_{ca} = \frac{\mathbf{f}^t_{ca}}{CA} = \frac{14}{0.68} = \underline{20.59 \text{ rad/s}^2}$$

counter-clockwise

***Example* 3.16** *The dimensions of a four-link mechanism are as under [Fig. 3.23(a)]:*

AB = 35 mm, BC = 40 mm, CD = 45 mm and AD = 70 mm. At the instant when ∠DAB= 75°, the link AB rotates with an angular velocity of 10 rad/s in the counter-clockwise direction. Given the coincident points P attached to the link CD and Q attached to the link BC such that BQ = 30 mm and CQ = 20 mm. Determine the acceleration of P relative to Q (or the link BC).

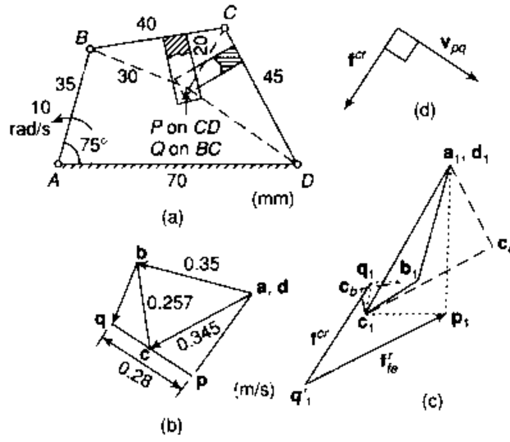


Fig. 3.23

Solution $v_b = 10 \times 0.035 = 0.35$ m/s

Complete the velocity diagram as shown in Fig. 3.23(b). Point Q can be located by drawing the velocity image of the triangle BCQ and P by drawing the velocity image of the triangle DCP.

For acceleration diagram, we have,

Acc. of C rel. to A = Acc. of C rel. to B + Acc. of B rel. to A

$$\mathbf{f}_{ca} = \mathbf{f}_{cb} + \mathbf{f}_{ba}$$

or $$\mathbf{f}_{cd} = \mathbf{f}_{ba} + \mathbf{f}_{cb}$$

or $$\mathbf{d}_1\mathbf{c}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{c}_1$$

Writing in terms of components,

$\therefore$ $$\mathbf{f}^c_{cd} + \mathbf{f}^t_{cd} = \mathbf{f}^c_{ba} + \mathbf{f}^c_{cb} + \mathbf{f}^t_{cb}$$

or $$\mathbf{d}_1\mathbf{c}_d + \mathbf{c}_d\mathbf{c}_1 = \mathbf{a}_1\mathbf{b}_1 + \mathbf{b}_1\mathbf{c}_b + \mathbf{c}_b\mathbf{c}_1$$

Set the following vector table (Table 18):

Draw the acceleration diagram $\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\mathbf{d}_1$ following the steps of Example 3.1. Locate the point $\mathbf{p}_1$ by drawing the acceleration image of the triangle DCP on the vector $\mathbf{c}_1\mathbf{d}_1$ and $\mathbf{q}_1$ by drawing the acceleration image of the triangle BCQ on the vector $\mathbf{b}_1\mathbf{c}_1$ [Fig.3.23(c)]

Now,

$\mathbf{f}_{pa}$ = Acc. of P rel. to Q + Acc. of Q rel. to A + Coriolis acceleration component

$$= \mathbf{f}_{pq} + \mathbf{f}_{qa} + \mathbf{f}^{cr}$$

In the above equation $\mathbf{f}_{pa}$ ($\mathbf{a}_1\mathbf{p}_1$) and $\mathbf{f}_{qa}$ ($\mathbf{a}_1\mathbf{q}_1$) are known and it is required to find $\mathbf{f}_{pq}$. Therefore, first we have to find $\mathbf{f}^{cr}$.

Table 18

SN	Vector	Magnitude (m/s²)	Direction	Sense
1.	$\mathbf{f}^c_{ba}$ or $\mathbf{a_1 b_1}$	$\frac{(\mathbf{ab})^2}{AB} = \frac{(0.35)^2}{0.035} = 3.5$	$\parallel AB$	$\to A$
2.	$\mathbf{f}^c_{cb}$ or $\mathbf{b_1 c_b}$	$\frac{(\mathbf{bc})^2}{BC} = \frac{(0.257)^2}{0.04} = 1.65$	$\parallel BC$	$\to B$
3.	$\mathbf{f}^t_{cb}$ or $\mathbf{c_b c_1}$	-	$\perp BC$	-
4.	$\mathbf{f}^c_{cd}$ or $\mathbf{d_1 c_d}$	$\frac{(\mathbf{dc})^2}{DC} = \frac{(0.345)^2}{0.045} = 2.65$	$\parallel DC$	$\to D$
5.	$\mathbf{f}^t_{cd}$ or $\mathbf{c_d c_1}$	-	$\perp DC$	-

$$*\mathbf{f}^{cr} = 2\omega_{bc}\, v_{pq}$$

$$= 2\frac{v_{bc}}{BC}\mathbf{qp}$$

$$= 2 \times \frac{0.257}{0.04} \times 0.28 = 3.6 \text{ m/s}^2$$

Its direction is given by as shown in Fig. 3.23(d) by rotating the vector $\mathbf{v}_{pq}$ in the direction of angular velocity of *BC* which is clockwise in this case.

f_{pq} is represented by vector $\mathbf{q_1 p_1}$ = <u>4.38 m/s²</u>

This is the acceleration of *P* relative to *Q* (or the link *BC*), i.e, the acceleration which an observer stationed on link *BC* would report as the acceleration of the point *P*.

3.7 ALGEBRAIC METHODS

Let us consider the same system of a plane moving body having its motion relative to a fixed coordinate system *xyz* as was taken in Section 2.11. A moving coordinate system *x'y'z'* is attached to this moving body as before (Fig. 3.24). Coordinates of the origin *A* of the moving system are known relative to the absolute reference system and the moving system has an angular velocity ω also.

To find the acceleration of *P*, a procedure similar to the one adopted for velocity is used here also.

Vector Approach

Equation 2.7 is

$$\mathbf{v}_p = \mathbf{v}_b + \mathbf{v}^R + \omega \times \mathbf{r}$$

Differentiating it to obtain the acceleration of *P*,

$$\dot{\mathbf{v}}_\mathbf{p} = \dot{\mathbf{v}}_\mathbf{b} + \dot{\mathbf{v}}^\mathbf{R} + \dot{\omega} \times \mathbf{r} + \omega \times \dot{\mathbf{r}}$$

where $\dot{\omega}$ is the angular acceleration of rotation of the moving system.

and $\dot{\mathbf{v}}^\mathbf{R}$ is obtained by differentiating $(\dot{x}'\mathbf{l} + \dot{y}'\mathbf{m} + \dot{z}'\mathbf{n})$,

i.e.,

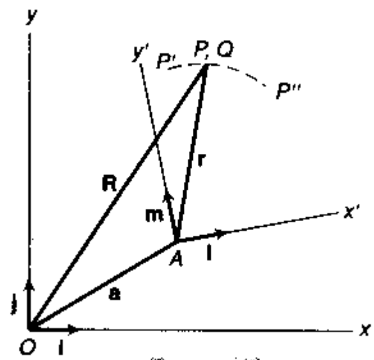


Fig. 3.24

$$\dot{v}^R = (\ddot{x}'l + \ddot{y}'m + \ddot{z}'n) + (\dot{x}'\dot{l} + \dot{y}'\dot{m} + \dot{z}'\dot{n})$$
$$= (\ddot{x}'l + \ddot{y}'m + \ddot{z}'n) + \omega(\dot{x}'l + \dot{y}'m + \dot{z}'n)$$
$$= f^R + \omega X v^R$$

$$\omega \times \dot{r} = \omega \times \frac{d}{dt}(x'l + y'm + z'n)$$
$$= \omega \times (\dot{x}'l + \dot{y}'m + \dot{z}'n) + \acute{E}(x'\dot{l} + y'\dot{m} + z'\dot{n})$$
$$= \omega \times v^R + \omega \times (\omega \times r)$$

But $\dot{\mathbf{v}}_p = \mathbf{f}_p$ and $\dot{\mathbf{v}}_b = \mathbf{f}_b$

Therefore,

$$\mathbf{f}_p = \mathbf{f}_b + (\mathbf{f}^R + \omega \times \mathbf{v}^R) + \dot{\omega} \times \mathbf{r} + [\omega \times \mathbf{v}^R + \omega \times (\omega \times \mathbf{r})]$$
$$= \mathbf{f}_b + \mathbf{f}^R + 2\omega \times \mathbf{v}^R + \dot{\omega} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r}) \quad \text{(i)}$$

Now absolute acceleration of Q, the coincident point may be written as

$$\mathbf{f}_{qa} = \mathbf{f}_{qb} + \mathbf{f}_{ba}$$
$$= \mathbf{f}_{ba} + \mathbf{f}_{qb}$$
$$= \mathbf{f}_b + \frac{\mathbf{d}}{\mathbf{dt}}(\omega \times \mathbf{r})$$
$$= \mathbf{f}_b + \dot{\omega} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r})$$

Thus equation (i) reduces to

$$\mathbf{f}_p = \mathbf{f}_{qa} + \mathbf{f}^R + 2\omega X \mathbf{v}^R$$
$$= \mathbf{f}_{qa} + \mathbf{f}^R + \mathbf{f}^C$$

where $\mathbf{f}_{qa}$ is the absolute acceleration of Q, $\mathbf{f}^R$ is the acceleration of P relative to the moving system or relative to Q, and $\mathbf{f}^C$ is known as the Coriolis component of acceleration.

The above equation may be written as

$$\mathbf{f}_{pa} = \mathbf{f}_{qa} + \mathbf{f}_{pq} + \mathbf{f}^c$$
$$\mathbf{f}_{pa} = \mathbf{f}_{pq} + \mathbf{f}_{qa} + \mathbf{f}^c$$

Acc. of P rel. to A = Acc. of P rel. to Q + Acc. of Q rel. to A + Coriolis Acc. (3.6)

Use of Complex Numbers

Equation 2.8 is

$$\mathbf{v} = \dot{r}e^{i\theta} + ir\dot{\theta}\, e^{i\theta}$$

Differentiating it with respect to time,

$$\mathbf{f} = (\ddot{r}e^{i\theta} + \dot{r}i\,\dot{\theta}e^{i\theta}) + (ir\ddot{\theta}\, e^{i\theta} + i\dot{r}\dot{\theta}\, e^{i\theta} + i^2 r\dot{\theta}\, e^{i\theta})$$
$$= (\ddot{r}e^{i\theta} - r\dot{\theta}^2)e^{i\theta} + i(r\ddot{\theta} + 2\dot{r}\dot{\theta})e^{i\theta} \quad (3.7)$$

The first part of this equation indicates the radial or *centripetal acceleration* and the second part, the *transverse acceleration* in polar coordinates.

$$\mathbf{f} = (f - \omega^2 r) + (r\alpha + 2\omega v)$$
$$= f + (r\alpha - \omega^2 r) + 2\omega v \quad (3.8)$$

= Acc. of P rel. to Q + Acc. of Q rel. to A + Coriolis acceleration component i.e., the same equation as before.

3.8 KLEIN'S CONSTRUCTION

In Klein's construction, the velocity and the acceleration diagrams are made on the configuration diagram itself. The line that represents the crank in the configuration diagram also represents the velocity and the acceleration of its moving end in the velocity and the acceleration diagrams respectively. For a slider-crank mechanism, the procedure to make the Klein's construction is described below.

Slider-Crank Mechanism

In Fig. 3.25, OAB represents the configuration of a slider-crank mechanism. Its velocity and acceleration diagrams are as shown in Figs. 3.4(b) and (c). Let r be the length of the crank OA.

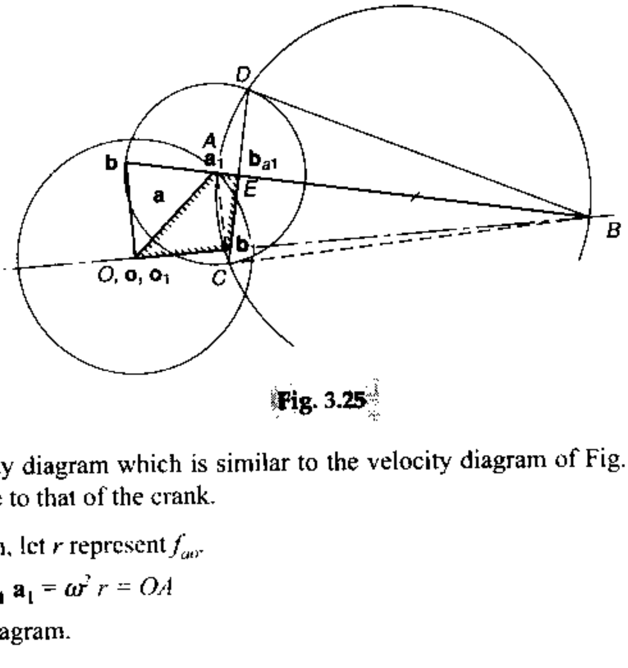


Fig. 3.25

Velocity Diagram For velocity diagram, let r represent v_{ao}, to some scale. Then for the velocity diagram, length $oa = \omega r = OA$.

From this, the scale for the velocity diagram is known.

Produce BA and draw a line perpendicular to OB through O. The intersection of the two lines locates the point **b**. the figure, **oab** is the velocity diagram which is similar to the velocity diagram of Fig. 3.4(b) rotated through 90° in a direction opposite to that of the crank.

Acceleration Diagram For acceleration diagram, let r represent f_{ao}.

$$\therefore \quad \mathbf{o_1 a_1} = \omega^2 r = OA$$

This provides the scale for the acceleration diagram.

Make the following construction:

1. Draw a circle with **ab** as the radius and **a** as the centre.
2. Draw another circle with AB as diameter.
3. Join the points of intersections C and D of the two circles. Let it meet OB at $\mathbf{b_1}$ and AB at E.

Then $\mathbf{o_1a_1b_ab_1}$ is the required acceleration diagram which is similar to the acceleration diagram of Fig. 3.4(c) rotated through 180°.

The proof is as follows:

Join AC and BC.

AEC and ABC are two right-angled triangles in which the angle CAB is common. Therefore, the triangles are similar.

$$\frac{AE}{AC} = \frac{AC}{AB} \quad \text{or} \quad AE = \frac{(AC)^2}{AB} \quad \text{or} \quad \mathbf{a_1\,b_a} = \frac{(\mathbf{ab})^2}{AB} = f_{ba}^c$$

Thus, this acceleration diagram has all the sides parallel to that of acceleration diagram of Fig. 3.4(c) and also has two sides $\mathbf{o_1a_1}$ and $\mathbf{a_1b_a}$ representing the corresponding magnitudes of the acceleration. Thus, the two diagrams are similar.

3.9 VELOCITY AND ACCELERATION FROM DISPLACEMENT–TIME CURVE

Sometimes, displacement–time data for a moving point in a mechanism are available and it is required to find the velocity and acceleration at various instants. This can be done easily by graphical differentiation that uses the following rules:

1. Velocity is the derivative of displacement with respect to time and is proportional to the slope of the tangent to the displacement–time curve for any instant.
2. Acceleration is the derivative of velocity with respect to time and is proportional to the slope of the tangent to the velocity–time curve for any instant.

Figure 3.26(a) shows a displacement–time curve of a point in a mechanism. At the point *C*, *LN* is the tangent where the points *L* and *N* are chosen arbitrarily.

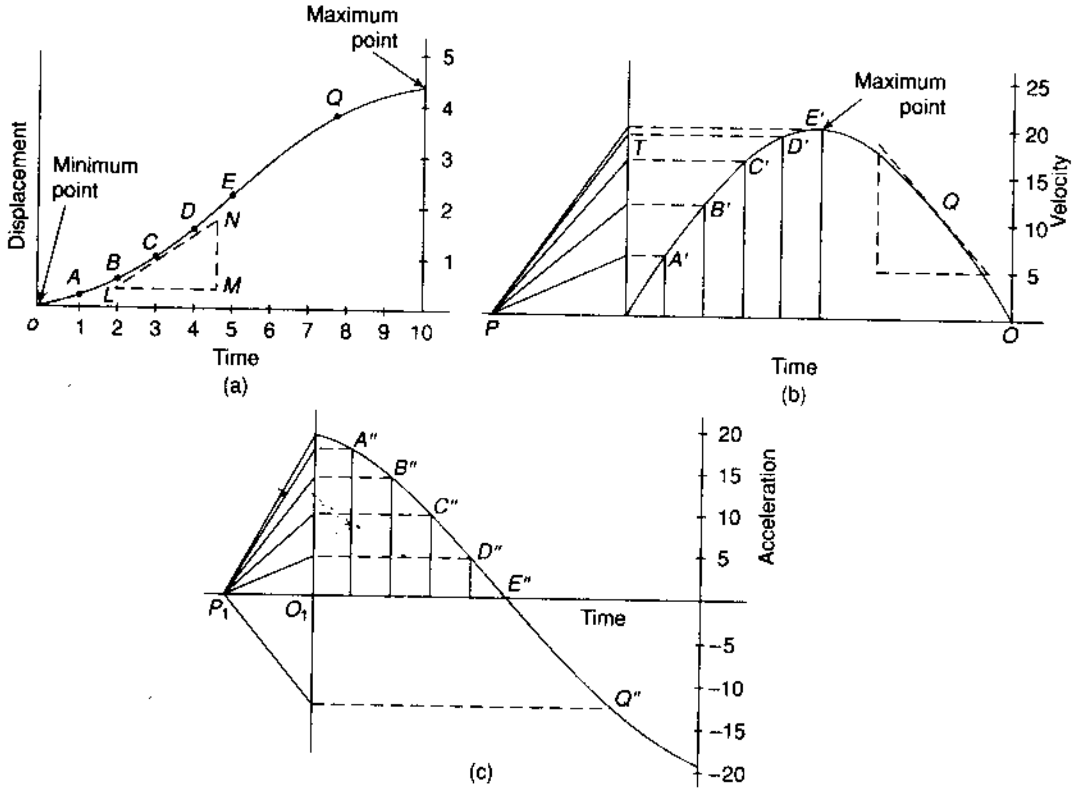


Fig. 3.26

Then
$$v_c = \frac{k_s\, NM}{k_t\, LM} \qquad \text{(i)}$$

where v_c = velocity at *C* and
k_s = displacement scale
k_t = time scale
and *NM*, *LM* = actual drawing distances

To plot the velocity–time curve, select a convenient point *P* (known as pole point) as shown in

Fig. 3.26 (b). Draw a line PT parallel to LN. Then it can be said that TO is the magnitude of the velocity at the point C to some scale which can be found as follows.

Let k_v = velocity scale

Then

$$v_c = k_v\, TO \qquad \text{(ii)}$$

From (i) and (ii),

$$k_v\, TO = \frac{k_s\, NM}{k_t\, LM}$$

$$= \frac{k_s\, TO}{k_t\, PO}$$

or

$$k_v = \frac{k_s}{k_t} \cdot \frac{1}{PO}$$

Thus, the scale of k_v is known.

- *Alternatively, the velocity scale may be chosen first, and accordingly the point T may be marked for the velocity v_c and then drawing a line parallel to LN will locate the pole point P.*

Select more points on the displacement–time curve and draw tangents to the curve. From the pole point P, draw lines parallel to these tangents meeting the Y-axis. Project the points obtained on this axis to the corresponding ordinates. Complete the velocity–time curve using a *french* curve.

In the same way, the acceleration–time curve can be drawn by taking another pole point P_1 for that [Fig. 3.26 (c)]. The acceleration scale k_f will be given by

$$k_f = \frac{k_v}{k_t} \cdot \frac{1}{P_1O_1}$$

Note that the derivative is

1. positive if a curve rises and is negative if it falls
2. zero at a maximum or minimum point on a curve
3. numerically maximum (positive or negative) at an inflection point (a point where the curvature changes on the curve)

3.10 CENTRE OF CURVATURE

In using the acceleration vector equations, it is necessary to carefully identify a point whose centre of curvature is known so that the radius of curvature of its locus is known which is needed to calculate the normal component of acceleration. It will be interesting and convenient if any arbitrary point is used in finding this component if its radius of curvature could be calculated. In the following sections, some methods are presented to find the same.

In a planar motion, when a rigid body moves relative to another, an arbitrary chosen point on the first body traces a path relative to a coordinate system fixed to the second body. For example, if two bodies p and q are in relative motion then a point A on the body p traces a path relative to the coordinate system fixed to the body q. At any instant, the point A may be assumed to move in a curve and thus has a centre of curvature A' in the body q. Considering the inversion of this motion, the point A' in the body q also moves in a curve relative to the body p with its centre of curvature at A. Thus, each point acts as the centre of curvature of the locus of the other. In the four-bar linkage of Fig. 3.2(a), A on the fixed link 1 is the centre of curvature of B on the moving coupler 3. Then considering the inversion, i.e., assuming link 3 to be fixed and releasing the fixed link 1, B on the link 3 is also the centre of curvature of the point A on the link 1. The two points are known as the *conjugates* of each other. The distance between them is called the radius of curvature of either locus.

3.11 HARTMANN CONSTRUCTION

The Hartmann construction is a graphical method to find the location of the centre of curvature of the locus of a point on a moving body. Let there be two bodies having a relative planar motion between them. Consider two curvatures of the two actual centrodes (Section 2.16) in the region near the point of contact at the instant. Let a circle with centre O' represent the circle corresponding to the curvature of fixed centrode and O, the centre of circle corresponding to the curvature of the moving centrode (Fig. 3.27). For the sake of convenience, the two circles may be called the fixed and the moving centrodes. Let I be the point of contact of the two centrodes which is also the instantaneous centre. The centrode tangent and the centrode normal are also shown in the figure.

Let the moving centrode roll on the fixed centrode with angular velocity ω. Then as I is also the instantaneous centre, the velocity of the point O is

$$v_o = \omega . OI$$

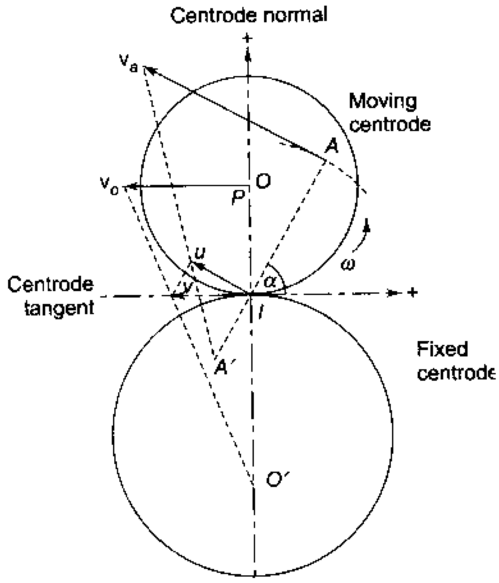


Fig. 3.27

As the motion of moving centrode advances, the point of contact P moves along with some velocity v. Since at any instant, the line joining O with O' must pass through P, the velocity of P must be given by

$$v = \frac{IO'}{OO'} v_o$$

The velocity of any arbitrary point A on the moving centrode, i.e., a point on the coupler whose conjugate point is to be found is given by, $v_a = \omega . AI$.

To find the conjugate point of the point A, the Hartmann construction is as follows:

1. Take a vector representing the velocity $\mathbf{v}_0$ of the point O by drawing a line perpendicular to OI to a suitable scale. Also, take a vector representing $\mathbf{v}_a$, the velocity of A by taking a line perpendicular to AI and drawn to the same scale.
2. Draw the velocity vector $\mathbf{v}$ to indicate the velocity of the point I by drawing a line parallel to $\mathbf{v}_0$ and intersecting with the line joining O' with the end point of the vector $\mathbf{v}_0$.
3. Take a component of $\mathbf{v}$ parallel to $\mathbf{v}_a$. Let this vector be called $\mathbf{u}$.
4. Join end points of the vectors $\mathbf{v}_a$ and $\mathbf{u}$. Then the intersection of this line with AI provides the requisite conjugate point A', giving the radius of curvature of the locus of A as AA'.

3.12 EULER–SAVARY EQUATION

An analytical expression known as the *Euler–Savary equation* for the location of the conjugate point of A is derived as follows:

If α is the angle between the centrode tangent and the line AP (Fig. 3.27), then as $v = \dfrac{IO'}{OO'} v_o$,

$$u = v \sin\alpha = \frac{IO'}{OO'} v_o . \sin\alpha = \frac{IO'}{OO'}.(\omega.OP)\sin\alpha = \frac{IO'.OI}{OO'}.\omega.\sin\alpha \qquad \text{(i)}$$

Also,

$$u = \frac{IA'}{AA'} v_a = \frac{IA'}{AA'}.(\omega.AI) = \frac{IA'.AI}{AA'}.\omega \qquad \text{(ii)}$$

From (i) and (ii), $\dfrac{IO'.OI}{OO'}.\omega.\sin\alpha = \dfrac{IA'.AI}{AA'}.\omega$

or

$$\frac{AA'}{AI.IA'}\sin\alpha = \frac{OO'}{OI.IO'}$$

or

$$\left(\frac{AI}{AI.IA'} + \frac{IA'}{AI.IA'}\right)\sin\alpha = \frac{OI}{OI.IO'} + \frac{IO'}{OI.IO'}$$

or

$$\left(\frac{1}{IA'} + \frac{1}{AI}\right)\sin\alpha = \frac{1}{IO'} + \frac{1}{OI}$$

or

$$\left(\frac{1}{AI} - \frac{1}{A'I}\right)\sin\alpha = \frac{1}{OI} - \frac{1}{O'I} \qquad \text{(iii)}$$

This is known as one form of the Euler–Savary equation. This is useful to locate the conjugate point A' of the point A when the radii of curvature of the two centrodes are known.

For any other point B at an angle β with the centrode tangent whose conjugate point is B' (Fig. 3.28), the above equation may be written as

$$\left(\frac{1}{BI} - \frac{1}{B'I}\right)\sin\beta = \frac{1}{OI} - \frac{1}{O'I}$$

Let this point be a particular point in the moving centrode such that it satisfies the equation

$$\frac{\sin\beta}{BI} = \frac{1}{OI} - \frac{1}{O'I}$$

This means that the term $1/B'I$ is zero which indicates that the point B is such that its conjugate point lies at infinity on the line joining BI.

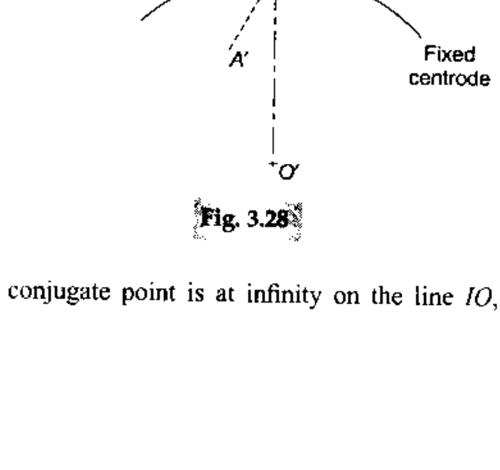


Fig. 3.28

Similarly, for a point P on the centrode normal whose conjugate point is at infinity on the line IO, $\dfrac{1}{PI} = \dfrac{1}{OI} - \dfrac{1}{O'I}$ as angle β is 90° and $\sin\beta$ is 1.

This also indicates that $PI = BI/\sin\beta$. The point P is known as the *inflection pole*.

Thus, to find more points whose conjugate points are at infinity, the equation $PI = BI/\sin\beta$ or $BI = PI/\sin\beta$ must be satisfied. This equation defines a circle whose diameter is IP as shown in Fig. 3.28. The circle is known as the *inflection circle*. Each point on this circle has an infinite radius of curvature at the instant and its conjugate point lies at infinity.

Thus, on the line AI, the point A'' intersecting the circle indicates that its conjugate point is at infinity and thus,

$$\frac{\sin\alpha}{A''I} = \frac{1}{OI} - \frac{1}{O'I} \tag{iv}$$

From (iii) and (iv), $\frac{1}{A''I} = \frac{1}{AI} - \frac{1}{A'I}$ or $\frac{1}{A''I} = \frac{A'I - AI}{AI.A'I}$ or $\frac{1}{A''I} = \frac{A'I + IA}{AI.A'I}$

or $\quad AI.A'I = A''I.A'A$

$\quad AI(A'A - IA) = (AI - AA'')A'A$

or $\quad AI.A'A - AI.IA = AI.A'A - AA''.A'A$

or $\quad AI.AI = AA''.AA'$

or $\quad AI^2 = AA''.AA' \qquad (3.9)$

This is the second form of the Euler–Savary equation and is more useful than the first form as this does not require knowing the curvatures of the two centrodes. However, it requires drawing the inflection circle which can easily be drawn.

In applying the above equation, AA' and AA'' are to lie on the same side of A.

Example 3.17 ***A slider-crank mechanism is shown in Fig. 3.29(a). The dimensions are:***
$OA = 20$ mm, $AB = 25$ mm, $AD = 10$ mm and $DC = 10$ mm.
Draw the inflection circle for the motion of the coupler and find the instantaneous radius of curvature of the path of the coupler point C.

Solution Locate the instantaneous centre of I at the intersection of OA and a line perpendicular to the direction of motion of the slider [Fig. 3.29(b)]. Apart from I, the point B also lies on the inflection circle as its centre of curvature is at infinity. One more point is needed to draw the inflection circle which can be obtained as follows:

As O is the centre of curvature of the point A, extend AO to A'' such that

$$AA'' = \frac{AI^2}{AO} = \frac{26.7^2}{20} = 35.6 \text{ mm}$$ (on measurement $AI = 26.7$ mm) (Eq. 3.9)

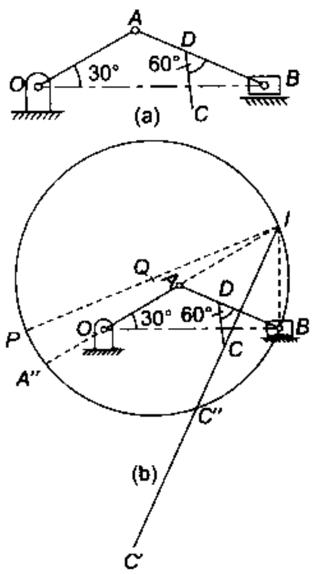


Fig. 3.29

Locate A'' as shown in the figure on the same side of A as A'. Thus, A'' is a point whose centre of curvature is at infinity.

Now draw a circle passing through points I, B and A'' by taking right bisectors of IB and BA'' (not shown in the figure) intersecting at the centre Q of the circle.

Diameter of the inflection circle, $IP = 62.5$ mm

To find the centre of curvature of the point C on the coupler, join IC intersecting the inflection circle at C''. Then C'' is a point having centre of curvature at infinity as this point lies on the inflection circle. Locate a point on IC or its extension such that

$$CC' = \frac{CI^2}{CC''} = \frac{29.9^2}{15.9} = 52.9 \text{ mm}$$

Locate C' as shown in the figure on the same side of C as C''. Then C' is the requisite centre of curvature of the point C.

3.13 BOBILLIER CONSTRUCTION

This is another graphical method by which inflection circle can be drawn without requiring the curvatures of the centrodes.

Let A and B be two points on the moving body which are not collinear with I (Fig. 3.30). Let A' and B' be their conjugate points respectively at the instant. Join AB and $A'B'$ and let their intersection be at Q. Then the line passing through I and Q is known as the *collineation axis*. This axis is specific for the two rays AA' and BB' and for another set of points A and B. Even on these rays, Q will have a different location and thus a different collineation axis.

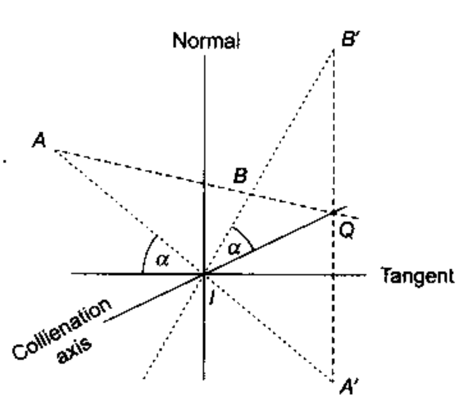
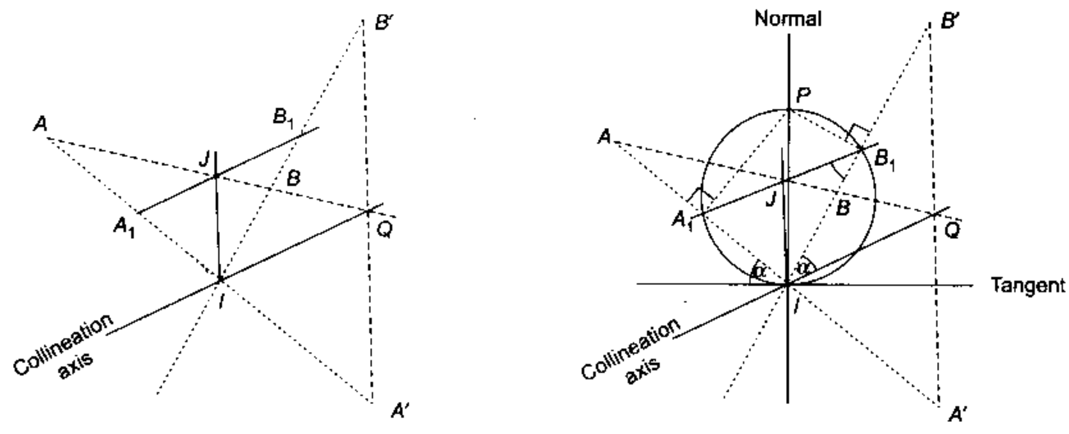


Fig. 3.30

Bobillier theorem It states that *the angle subtended by one of the rays (AA′ or BB′) with the centrode tangent is equal to negative of the angle subtended by the other ray with the collineation axis.*

In Fig. 3.30, the ray AA' subtends angle α with the centrode tangent and the ray BB' subtends the same negative angle with the collineation axis.

Proof



(a) (b)

Fig. 3.31

Let A and A', and B and B' be the known pairs of conjugate points [Fig. 3.31(a)].

Make the following construction:

1. Locate the point I, the instantaneous centre of velocity at the intersection of two rays AA' and BB'.
2. Locate the point Q at the intersection of rays AB and $A'B'$.
3. Join IQ to obtain the collineation axis.
4. Draw a line parallel to $A'B'$ intersecting AB at J.
5. Draw a line parallel to IQ through J intersecting AA' and BB' at A_1 and B_1 respectively.
6. Draw a circle passing through I, A_1 and B_1 (Fig. 3.31b). A convenient way of drawing the circle is by drawing $A_1P \perp AA'$ and $B_1P \perp BB'$ intersecting two perpendicular lines at P. Now IP is the diameter of the inflection circle as it subtends a 90° angle at points A_1 and B_1 indicating that A_1 and B_1 are the points in the semicircles with diameter IP. Thus, P is the inflection pole. Draw the circle with IP as the diameter.
7. As IP is also the centrode normal, draw the centrode tangent as shown in the figure.

Let α be the angle which IA_1 subtends with the centrode tangent. Now, arc IA_1 is inscribed by the chord IA_1 which is at an angle α with the centrode tangent and subtends the angle IB_1A_1 at the circumference of the inflection circle. Therefore, the angle IB_1A_1 is also equal to α. As A_1B_1 is parallel to PQ and is intersected by IB', the angle IB_1A_1 is also equal to the angle QIB_1 i.e., equal to α. Thus, the angle subtended by one of the rays with the centrode tangent is equal to the negative of the angle subtended by the other ray with the collineation axis. Thus the construction satisfies the Bobillier theorem.

Method to find a conjugate point of another arbitrary point

Let the inflection circle be drawn and the centrode tangent and normal be known and it is required to find the conjugate point of C (Fig. 3.32). The point P is the inflection pole, i.e., its conjugate point P' lies at infinity and thus the ray PP' is perpendicular to the tangent to the centrode tangent. This suggestes that according to the Bobillier theorem, the other ray CC' will be perpendicular to the collineation axis. But as the point C' must lie on IC, the collineation axis can be drawn by drawing a line perpendicular to IC at I. Since Q is a point of intersection of two rays PC and $P'C'$, it can be located at the intersection of PC and the collineation axis. Now as Q also lies on $P'C'$, joining of $P'Q$ means a line parallel to IP, the intersection of this line with IC locates the point C'.

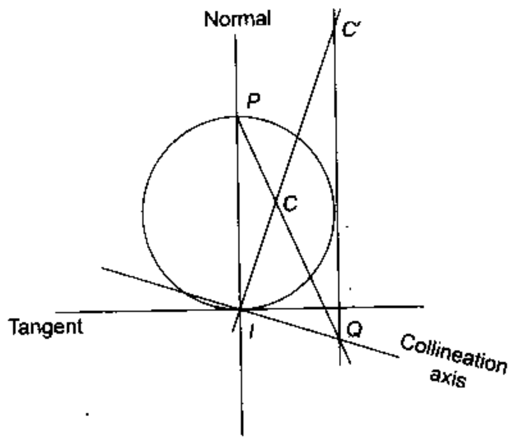


Fig. 3.32

Thus, the procedure to find the conjugate point C' of any arbitrary point C is as follows:

- Draw the collineation axis by drawing a line perpendicular to IC through I. Locate Q at the intersection of PC with the collineation axis.
- Draw a line parallel to IP through Q intersecting the line IC at C', the requisite conjugate point of C.

Example 3.18 *Use the Bobillier theorem to determine the centre of curvature of the coupler curve of the point E of the four-bar mechanism shown in Fig. 3.33(a). The dimensions are AD = AB = 60 mm, BC = CD = 25 mm. AD is the fixed link and E is the midpoint of BC.*

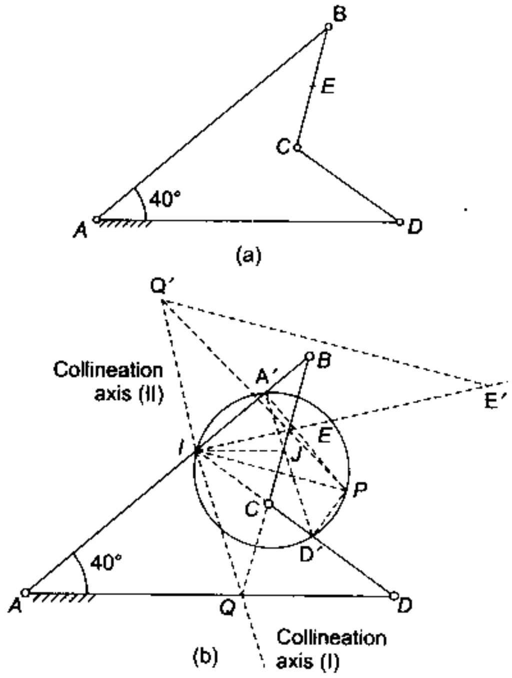


Fig. 3.33

Solution Proceed as follows:

1. Locate points I and Q. Join IQ which is the collineation axis [Fig. 3.33(b)].
2. Draw IJ parallel to AD intersecting BC at J. Draw $A'D'$ through J parallel to IQ and obtain points A' and D' on AB and DC respectively.
3. Through A' draw a perpendicular to AB and through D' draw a perpendicular to DC. Let these perpendiculars intersect at the inflection point P. Draw the inflection circle with IP as the diameter.
4. To find the conjugate point of E, draw the ray IE. Then obtain the new collineation axis by drawing a line perpendicular to IE through I. Locate Q' at the intersection of PE with the collineation axis.
5. Draw a line parallel to IP through Q' intersecting the line IE at E', the requisite conjugate point of E.

On measurement, $EE' = 33$ mm

3.14 CUBIC OF STATIONARY CURVATURE

Usually, the coupler curve (the locus or path of a point on the coupler) is a sixth-order curve whose radius of curvature changes continuously. However, it is observed that in certain situations, the path has a stationary curvature. Thus, if R is the radius of curvature and s is the distance traveled along the path, then $dR/ds = 0$ indicates a stationary curvature of the curve. The locus of all such points on the coupler which have stationary curvature at the instant is known as the *cubic of the stationary curvature* or the *circling-point curve*. Note that the stationary curvature does not mean only a constant radius, but also that the continuously varying radius passes through a maximum or minimum value.

Graphical Method

Let the four-link mechanism be $ABCD$ as shown in Fig. 3.34(a) in which AD is the fixed link. Now, as the link AB can rotate about A only, therefore, A is also the conjugate of B with a constant radius of curvature AB. Thus, A lies on the cubic curve. Similarly, C also lies on the cubic as it has a constant radius of curvature CC'.

Now, adopt the following procedure:

1. Locate points I and Q as usual. Join IQ which is the collineation axis.
2. Let the angle subtended by the ray AB with the collineation axis be α. The same angle is subtended by the other ray CD with the centrode tangent at the point I in the opposite direction. Thus, make angle DIT equal to α with IA in the counter-clockwise direction as the angle made by AB with the collineation axis is clockwise. Then, IT is the centrode tangent.

3. Draw a line IN perpendicular to IT. Then IN is the centrode normal.
4. Draw a line perpendicular to IA at B intersecting IT and IN at B_t and B_n respectively. Through B_t and B_n, draw lines parallel to IN and IT respectively intersecting at B_1.
5. Repeat the step 4 by drawing a perpendicular to ID at C and obtain the point C_1. Draw a line joining B_1C_1 which is an auxiliary line used to obtain other points on the cubic. Let B_1C_1 intersect the centrode tangent at L and the centrode normal at M.
6. Choose any point G_1 on the line B_1C_1 [Fig. 3.34(b)] and draw lines parallel to the tangent and normal and intersecting these at G_t and G_n. Draw $IG \perp G_t G_n$. Then G is another point on the cubic of the stationary curve. Similarly, choose more points (such as H_1) on the line B_1C_1 and obtain more points lying on the curve. Draw a smooth curve passing through these points which is the required curve of the cubic of stationary curvature.

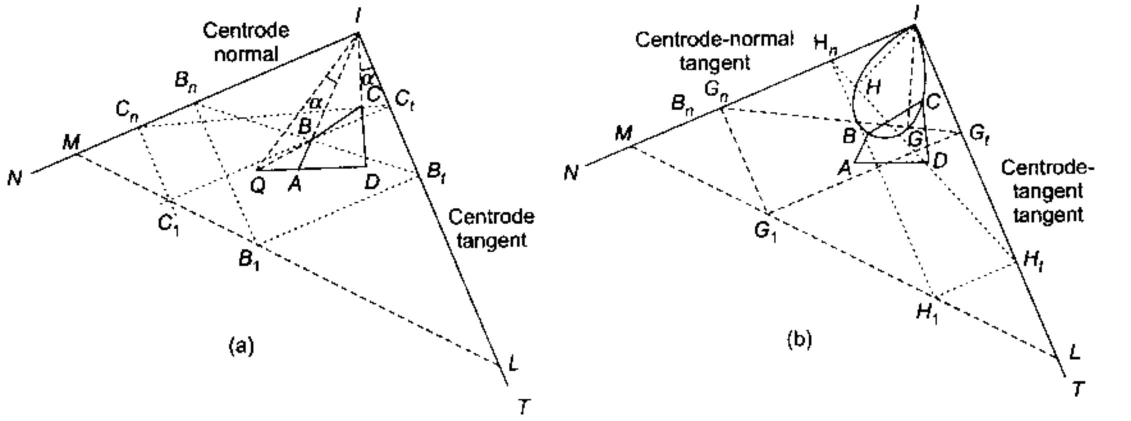


Fig. 3.34

Note that there are two tangents at I to the cubic of stationary curvature known as the centrode-normal tangent and the centrode-tangent tangent.

Radius of curvature of the cubic at the centrode-normal tangent = $IL/2$

Radius of curvature of the cubic at the centrode-tangent tangent = $IM/2$

The equation of the cubic of stationary curvature can be written as

$$\frac{1}{A_1 \sin\varphi} + \frac{1}{A_2 \sin\varphi} = \frac{1}{r} \tag{3.10}$$

where r is the distance of the point considered on the cubic from the instantaneous centre I at an angle ϕ subtended by a line joining the point and I with the centrode tangent. A_1 and A_2 are constants and can be found using any two known points lying on the cubic such as C and D.

Ball's Point A point at the intersection of the cubic with the inflection circle is known as Ball's point. It traces an approximately straight path as it has a stationary curvature of infinity.

Summary

1. Acceleration is the derivative of velocity with respect to time and is proportional to the slope of the tangent to the velocity–time curve for any instant.
2. The rate of change of velocity in the tangential direction of the motion of a particle is known as the *tangential acceleration*.

3. The rate of change of velocity along the radial direction is known as the *centripetal* or *radial acceleration*, the direction being towards the centre of rotation.
4. The angular acceleration of a link about one extremity is the same in magnitude and direction as the angular acceleration about the other and is found by dividing the tangential acceleration with the length of the link.
5. *Acceleration images* are helpful to find the accelerations of offset points of the links. The acceleration image of a link is obtained in the same manner as a velocity image.
6. Acceleration of a point on a link relative to a coincident point on a moving link is the sum of absolute acceleration of the coincident point, acceleration of the point relative to coincident point and the *Coriolis* acceleration.
7. The *Hartmann construction* is a graphical method to find the location of the centre of curvature of the locus of a point on the moving body.
8. The *Euler–Savary equation* is expressed as $AI^2 = AA''.\ AA'$
9. The Bobillier construction is another graphical method by which an inflection circle can be drawn without requiring the curvatures of the centrodes.
10. The *Bobillier theorem* states that *the angle subtended by one of the rays (AA′ or BB′) with the centrode tangent is equal to the negative of the angle subtended by the other ray with the collineation axis.*
11. The locus of all such points on the coupler which have stationary curvature at the instant is known as the *cubic of the stationary curvature* or the *circling-point curve*.

Exercises

1. What are centripetal and tangential components of acceleration? When do they occur? How are they determined?
2. Describe the procedure to draw velocity and acceleration diagrams of a four-link mechanism. In what way are the angular accelerations of the output link and the coupler found?
3. What is an acceleration image? How is it helpful in determining the accelerations of offset points on a link?
4. What is the Coriolis acceleration component? In which cases does it occur? How is it determined?
5. Explain the procedure to construct Klein's construction to determine the velocity and acceleration of a slider-crank mechanism.
6. Explain the term conjugates in relation to two points on two plain bodies.
7. Explain the Hartmann construction to find the location of the centre of curvature of the locus of a point on a moving body.
8. What is Euler–Savary equation? What are its two forms? Explain how these are used to find the location of conjugate points.
9. Use the Bobillier theorem to show that the inflection circle can be drawn without requiring the curvatures of the centrodes.
10. Define the term *cubic of the stationary curvature*. Explain one graphical method to draw it.
11. A crank and rocker mechanism *ABCD* has the following dimensions:
AB = 0.75 m, BC = 1.25 m, CD = 1 m, AD = 1.5 m.
BE = 437.5 mm, CE = 87.5 mm and CF = 500 mm
E and F are two points on the coupler link BC. AD is the fixed link. BEC is read clockwise and F lies on BC produced. Crank AB has an angular velocity of 20.94 rad/s counter-clockwise and a deceleration of 280 rad/s² at the instant $\angle DAB = 60°$. Find the
(i) instantaneous linear acceleration of C, E and F
(ii) instantaneous angular velocities and accelerations of links BC and CD
[(i) 166 m/s², 330 m/s², 161 m/s² (ii) ω_{bc} = 5.92 rad/s cw, ω_{cd} = 11.5 rad/s ccw, ω_{bc} = 229 rad/s² ccw, α_{cd} = 100 rad/s² ccw]

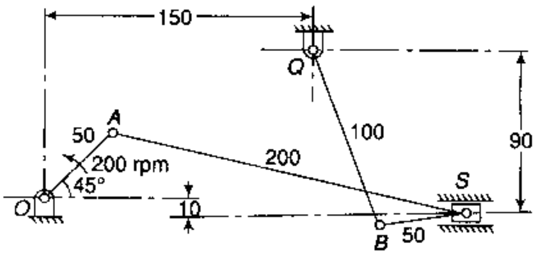


Fig. 3.35

12. Figure 3.35 shows a mechanism in which O and Q are the fixed centres. Determine the acceleration of the slider S and the angular acceleration of the link BQ for the given configuration.
(14.5 m/s² towards left; 114 rad/s² cw)

13. In a simple steam engine, the lengths of the crank and the connecting rod are 100 mm and 400 mm respectively. The weight of the connecting rod is 50 kg and its centre of mass is 220 mm from the cross-head centre. The radius of gyration about the centre of mass is 120 mm. If the engine speed is 300 rpm, determine for the position when the crank has turned 45° from the inner-dead centre,
    (i) the velocity and acceleration of the centre of mass of the connecting rod
    (ii) the angular velocity and acceleration of the rod
    (iii) the kinetic energy of the rod
    [(i) 2.7 m/s, 80 m/s² (ii) 5.7 rad/s, 173 rad/s² (iii) 194 N.m]
14. From the data of a reciprocating pump given in Example 2.4, find the linear acceleration of the cross-head *E* and the angular accelerations of the links *BCD* and *DE*.
    [9.25 m/s²; 60.8 rad/s²; 5.12 rad/s²]
15. Figure 3.36 shows a toggle mechanism in which the crank OA rotates at 120 rpm. Find the velocity and the acceleration of the slider at D.
    (0.17 m/s; 0.83 m/s²)

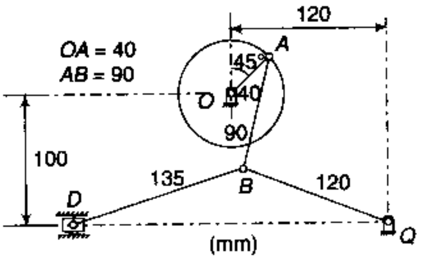


Fig. 3.36

16. In a crank and slotted-lever quick-return mechanism (Fig. 3.15a), the distance between the fixed centres *O* and *A* is 250 mm. Other lengths are: *OP* = 100 mm, *AR* = 400 mm, *RS* = 150 mm and ∠*AOP* = 120°. Uniform speed of the crank is 60 rpm clockwise. Line of stroke of the ram is perpendicular to *OA* and is 450 mm above *A*. Calculate the velocity and the acceleration of the ram *S*. **(0.64 m/s; 1.55 m/s²)**
17. For the inverted slider-crank mechanism of Example 2.13, determine the angular acceleration of the link *QR*. **(358 rad/s)**
18. In the pump mechanism shown in Fig. 3.22(a), the crank *OA* is 50 mm long and the piston rod *AC* is 150 mm long. The lengths *OQ* and *CQ* are 250 mm and 80 mm respectively. The crank rotates at 300 rpm in the clockwise direction. Determine the
    (a) velocity of the piston relative to walls
    (b) angular velocities of rod *AC* and the cylinder
    (c) sliding acceleration of the piston relative to cylinder
    (d) velocity of piston (absolute)
    (e) angular acceleration of the piston rod *BC*
    [(a) 1.51 m/s (b) 2.06 rad/s ccw of both, rod *AC* and cylinder (c) 16 m/s² (d) 1.5 m/s (e) 239 rad/s² ccw]

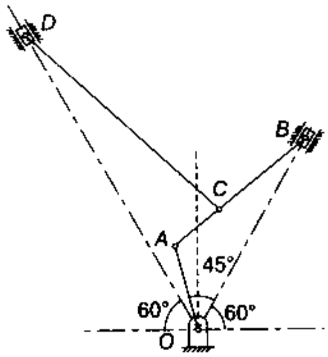


Fig. 3.37

19. In the mechanism shown in Fig. 3.37, the crank *OA* drives the sliders *B* and *D* in straight paths through connecting links *AB* and *CD*. The lengths of the links are *OA* = 150 mm, *AB* = 300 mm, *AC* = 100 mm, *CD* = 450 mm. *OA* rotates at 60 rpm clockwise and at the instant has angular retardation of 16 rad/s². Determine (i) the velocity and acceleration of sliders *B* and *D*, and (ii) the angular velocity and angular acceleration of link *CD*.
    (0.92 m/s, 0.31 m/s, 5.55 m/s², 5.49 m/s²; 2.07 rad/s, 6.53 rad/s²)

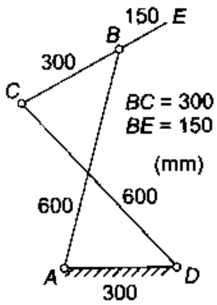


Fig. 3.38

20. For the motion of the coupler relative to the fixed link of the four-link mechanism as shown in Fig. 3.38, locate the position of the centre of curvature of the point *E* using the Bobillier theorem.

4 COMPUTER-AIDED ANALYSIS OF MECHANISMS

Introduction

The analyses of the velocity and the acceleration, given in chapters 2 and 3, depend upon the graphical approach and are suitable for finding out the velocity and the acceleration of the links of a mechanism in one or two positions of the crank. However, if it is required to find these values at various configurations of the mechanism or to find the maximum values of maximum velocity or acceleration, it is not convenient to draw velocity and acceleration diagrams again and again. In that case, analytical expressions for the displacement, velocity and acceleration in terms of the general parameters are derived. A desk-calculator or digital computer facilitates the calculation work.

4.1 FOUR-LINK MECHANISM

Displacement Analysis

A four-link mechanism shown in Fig. 4.1 is in equilibrium. a, b, c and d represent the magnitudes of the links AB, BC, CD and DA respectively. θ, β and φ are the angles of AB, BC and DC respectively with the x-axis (taken along AD). AD is the fixed link. AB is taken as the input link whereas DC as the output link.

As in any configuration of the mechanism, the figure must enclose, the links of the mechanism can be considered as vectors. Thus, vector displacement relationships can be derived as follows.

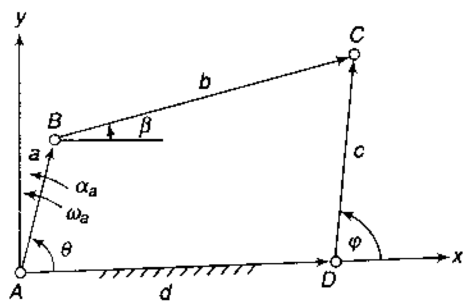


Fig. 4.1

Displacement along x-axis

$$a \cos \theta + b \cos \beta = d + c \cos \phi \tag{4.1}$$

(The equation is valid for $\angle\phi$ more than 90° also.)

or

$$b \cos \beta = c \cos \varphi - a \cos \theta + d$$

or

$$(b \cos \beta)^2 = (c \cos \varphi - a \cos \theta + d)^2$$

$$= c^2 \cos^2 \varphi + a^2 \cos^2 \theta + d^2 - 2ac \cos \theta \cos \varphi - 2ad \cos \theta + 2cd \cos \varphi \tag{4.2}$$

Displacement along *y*-axis

$$a \sin\theta + b \sin\beta = c \sin\phi \quad (4.3)$$

or $$b \sin\beta = c \sin\varphi - a \sin\theta$$

or $$(b \sin\beta)^2 = (c \sin\varphi - a \sin\theta)^2$$

$$= c^2 \sin^2\varphi + a^2 \sin^2\theta - 2ac \sin\theta \sin\varphi \quad (4.4)$$

Adding equations (4.2) and (4.4),

$$b^2 = c^2 + a^2 + d^2 - 2ac \cos\theta \cos\varphi - 2ad \cos\theta + 2cd \cos\varphi - 2ac \sin\theta \sin\varphi \quad (4.5)$$

Put

$$a^2 - b^2 + c^2 + d^2 = 2k$$

Then,

$$2cd \cos\varphi - 2ac \cos\theta \cos\varphi - 2ac \sin\theta \sin\varphi - 2ad \cos\theta + 2k = 0$$

or

$$cd \cos\varphi - ac \cos\theta \cos\varphi - ac \sin\theta \sin\varphi - ac \cos\theta + k = 0 \quad (4.6)$$

From trigonometric identities,

$$\sin\varphi = \frac{2\tan\left(\frac{\varphi}{2}\right)}{1 + \tan^2\left(\frac{\varphi}{2}\right)}$$

$$\cos\varphi = \frac{1 - \tan^2\left(\frac{\varphi}{2}\right)}{1 + \tan^2\left(\frac{\varphi}{2}\right)}$$

$$cd\left[\frac{1 - \tan^2(\varphi/2)}{1 + \tan^2(\varphi/2)}\right] - ac\cos\theta\left[\frac{1 - \tan^2(\varphi/2)}{1 + \tan^2(\varphi/2)}\right] - ac\sin\theta\left[\frac{2\tan(\varphi/2)}{1 + \tan^2(\varphi/2)}\right] - ad\cos\theta + k = 0$$

Multiplying throughout by $\left[1 + \tan^2\left(\frac{\varphi}{2}\right)\right]$

$$cd - cd\tan^2\left(\frac{\varphi}{2}\right) - ac\cos\theta + ac\cos\theta\tan^2\left(\frac{\varphi}{2}\right) - 2ac\sin\theta\tan\left(\frac{\varphi}{2}\right)$$

$$- ad\cos\theta - ad\cos\theta\tan^2\left(\frac{\varphi}{2}\right) + k + k\tan^2\left(\frac{\varphi}{2}\right) = 0$$

$$[k - a(d - c)\cos\theta - cd]\tan^2\left(\frac{\varphi}{2}\right) + [-2ac\sin\theta]\tan\left(\frac{\varphi}{2}\right) + [k - a(d + c)\cos\theta + cd] = 0$$

or

$$A\tan^2\left(\frac{\varphi}{2}\right)+B\tan\left(\frac{\varphi}{2}\right)+C=0$$

where

$A = k - a\,(d - c)\cos\theta - cd$

$B = -\,2ac\sin\theta$

$C = k - a\,(d + c)\cos\theta + cd$

Equation (4.6) is a quadratic in $\tan\left(\frac{\varphi}{2}\right)$. Its two roots are

$$\tan\left(\frac{\varphi}{2}\right)=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

or

$$\varphi = 2\tan^{-1}\left[\frac{-B\pm\sqrt{B^2-4AC}}{2A}\right] \tag{4.7}$$

Thus, the position of the output link, given by angle φ, can be calculated if the magnitude of the links and the position of the input link are known, i.e., a, b, c, d and θ are known.

A relation between the coupler link position β and the input link position θ can also be found as below:

Equations (4.1) and (4.3) can be written as,

$$c\cos\varphi = a\cos\theta + b\cos\beta - d \tag{4.8}$$

$$c\sin\varphi = a\sin\theta + b\sin\beta \tag{4.9}$$

Squaring and adding the two equations,

$$c^2 = a^2 + b^2 + d^2 + 2ab\cos\theta\cos\beta - 2bd\cos\beta - 2ad\cos\theta + 2ab\sin\theta\sin\beta$$

Put $\quad a^2 + b^2 - c^2 + d^2 = 2k'$

$$-\,2bd\cos\beta + 2ab\cos\theta\cos\beta + 2ab\sin\theta\sin\beta - 2ad\cos\theta + 2k' = 0$$

$$-\,bd\cos\beta + ab\cos\theta\cos\beta + ab\sin\theta\sin\beta - ad\cos\theta + k' = 0 \tag{4.10}$$

Equation (4.10) is identical to Eq. 4.6 and can be obtained from the same by substituting β for φ, $-b$ for c and k' for k.

Thus, the solution of Eq. (4.10) will be,

$$\beta = 2\tan^{-1}\left[\frac{-E\pm\sqrt{E^2-4DF}}{2D}\right] \tag{4.11}$$

where $\quad D = k' - a(d + b)\cos\theta + bd$

$E = 2ab\sin\theta$

$F = k' - a(d - b)\cos\theta - bd$

β can also be found directly from relation (4.3) after calculating φ.

Velocity Analysis

Let ω_a, ω_b and ω_c be the angular velocities of the links *AB*, *BC* and *CD* respectively.
Rewriting Eq. (4.1),

$$a \cos \theta + b \cos \beta - c \cos \phi - d = 0 \tag{4.12}$$

Differentiating it with respect to time,

$$\frac{d}{dt}(a \cos\theta + b \cos\beta - c \cos\varphi - d) = 0$$

$$\frac{d}{d\theta}\frac{d\theta}{dt}(a \cos\theta) + \frac{d}{d\beta}\frac{d\beta}{dt}(b \cos\beta) - \frac{d}{d\varphi}\frac{d\varphi}{dt}(c \cos\varphi) - \frac{d}{dt}(d) = 0$$

$$\frac{d\theta}{dt}\frac{d}{d\theta}(a \cos\theta) + \frac{d\beta}{dt}\frac{d}{d\beta}(b \cos\beta) - \frac{d\varphi}{dt}\frac{d}{d\varphi}(c \cos\varphi) - 0 = 0 \qquad (d \text{ is constant})$$

$$-a\, \omega_a \sin\theta - b\omega_b \sin\beta + c\, \omega_c \sin\phi = 0 \tag{4.13}$$

Similarly, rewriting Eq. (4.3),

$$a \sin \theta + b \sin \beta - c \sin \phi = 0 \tag{4.14}$$

Differentiating it with respect to time,

$$a\, \omega_a \cos\theta + b\omega_b \cos\beta - c\, \omega_c \cos\varphi = 0 \tag{4.15}$$

Multiply Eq. (4.13) by $\cos\beta$ and Eq. (4.15) by $\sin\beta$ and add,

$$a\, \omega_a(\sin\beta \cos\theta - \cos\beta \sin\theta) - c\omega_c(\sin\beta \cos\varphi - \cos\beta \sin\varphi) = 0$$

or $\quad a\, \theta_a \sin(\beta - \theta) - c\omega_c \sin(\beta - \phi) = 0$

or

$$\omega_c = \frac{a\omega_\alpha \sin(\beta - \theta)}{c \sin(\beta - \varphi)} \tag{4.16}$$

Multiply Eq. (4.13) by $\cos\varphi$ and Eq. (4.15) by $\sin\varphi$ and add,

$$a\, \omega_a(\sin\varphi \cos\theta - \cos\varphi \sin\theta) + b\omega_b(\sin\varphi \cos\beta - \cos\varphi \sin\beta) = 0$$

or

$$a\, \omega_a \sin(\varphi - \theta) + b\omega_b \sin(\varphi - \beta) = 0$$

or

$$\omega_b = -\frac{a\omega_\alpha \sin(\varphi - \theta)}{b \sin(\varphi - \beta)} \tag{4.17}$$

Since a, b, c, θ, β, φ and ω_a are already known, ω_c and ω_b can be calculated from Eqs (4.16) and (4.17) respectively.

Acceleration Analysis

Let α_a, α_b, and α_c be the angular accelerations of the links a, b and c respectively.

Differentiating equations (4.13) and (4.15) with respect to time in the above manner or rewriting in the following form,

$$-a\,\omega_a \sin\omega_a t - b\omega_b \sin\omega_b t + c\,\omega_c \sin\omega_c t = 0 \tag{4.18}$$

$$a\,\omega_a \cos\omega_a t + b\omega_b \cos\omega_b t - c\,\omega_c \cos\omega_c t = 0 \tag{4.19}$$

Differentiating these equations with respect to time,

$$(-a\alpha_a \sin\theta - a\omega_a^2 \cos\theta) - (b\alpha_b \sin\beta - b\omega_b^2 \cos\beta) + (c\alpha_c \sin\varphi + c\omega_c^2 \cos\varphi) = 0 \tag{4.20}$$

$$(a\alpha_a \cos\theta - a\omega_a^2 \sin\theta) + (b\alpha_b \cos\beta - b\omega_b^2 \sin\beta) - (c\alpha_c \cos\varphi + c\omega_c^2 \sin\varphi) = 0 \tag{4.21}$$

where $\alpha_\alpha = \dfrac{d\omega_a}{dt}, \alpha_b = \dfrac{d\omega_b}{dt}$ and $\alpha_c = \dfrac{d\omega_c}{dt}$

Multiply Eq. (4.20) by cos φ and Eq. (4.21) by sin φ and add,

$$a\alpha_a(\sin\varphi\cos\theta - \cos\varphi\sin\theta) - a\omega_a^2(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$$
$$-b\alpha_b(\sin\beta\cos\varphi - \cos\beta\sin\varphi) - b\omega_b^2(\cos\beta\cos\varphi + \sin\beta\sin\varphi) + c\omega_c^2 = 0$$

or

$$a\alpha_a \sin(\varphi-\theta) - a\omega_a^2\cos(\varphi-\theta) - b\alpha_b \sin(\beta-\varphi) - b\omega_b^2\cos(\varphi-\beta) + c\omega_c^2 = 0$$

or

$$\alpha_b = \frac{a\alpha_a \sin(\varphi-\theta) - a\omega_a^2\cos(\varphi-\theta) - b\omega_b^2\cos(\varphi-\beta) + c\omega_c^2}{b\sin(\beta-\varphi)} \tag{4.22}$$

Multiply Eq. (4.20) by cos β and Eq. (4.21) by sin β and add,

$$a\alpha_a(\sin\beta\cos\theta - \cos\beta\sin\theta) - a\omega_a^2(\cos\beta\cos\theta + \sin\beta\sin\theta) - b\omega_b^2$$
$$+c\alpha_c(\sin\varphi\cos\beta - \cos\varphi\sin\beta) + c\omega_c^2(\cos\beta\cos\varphi + \sin\beta\sin\varphi) = 0$$

or

$$a\alpha_a \sin(\beta-\theta) - a\omega_a^2\cos(\beta-\theta) - b\omega_b^2 - c\alpha_c \sin(\beta-\varphi) + c\omega_c^2\cos(\beta-\varphi) = 0$$

or

$$\alpha_c = \frac{a\alpha_a \sin(\beta-\theta) - a\omega_a^2\cos(\beta-\theta) - b\omega_b^2 + c\omega_c^2\cos(\beta-\varphi)}{c\sin(\beta-\varphi)} \tag{4.23}$$

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
 int i,j,iht,th,theta,limit,ins;
 float a,b,c,d,vela,acca,thet,aa,bb,cc,bet1,bet2,betd1,
 betd2,num1,num2,phi1,ph1,unroot,undroot,pi,k,phh,phi2,
 ph2,vel2,dthet;
 float num[2],phi[2],ph[2],bet[2],betd[2],b1[2],b2[2],
 b3[2],b4[2],c1[2],c2[2],c3[2],c4[2],accc[2],accb[2],
 velb[2],velc[2];
 clrscr();

 printf("enter values a,b,c,d,vela,acca,theta,limit\n");
 scanf("%f%f%f%f%f%f%d%d",&a,&b,&c,&d,&vela,&acca,
 &theta,&limit);
 printf( "   thet  vela  acca    phi    beta ");
 printf( "   velc   velb   accc   accb \n");
 ins=1;
 if(vela==0 && acca>0)ins=0;

 pi=4*atan(1);
 iht=360/theta;
 if(vela>0 && acca==0){ins=0;iht=360/theta; }
 if(ins==1)iht=theta;
 dthet=pi*2/iht;
 if(vela==0 && acca>0)iht=iht+limit/theta;
  for(j=0;j<iht+1;j++)
  {
   if(j>(iht-360/theta-1) && ins==0)acca=0;
   thet=j*dthet;
   if(ins==1){j=iht; thet=theta*pi/180;}
   th=theta*j;
   if(ins==1)th=theta;
   k=(a*a-b*b+c*c+d*d)/2;
   aa=k-a*(d-c)*cos(thet)-c*d;
   bb=-2*a*c*sin(thet);
   cc=k-a*(d+c)*cos(thet)+c*d;
   unroot=bb*bb-4*aa*cc;
     if(unroot>0)
```

```
    {
      undroot=sqrt(unroot);
      num[0]=-bb+undroot;
      num[1]=-bb-undroot;
        for(i=0;i<2;i++)
        {
          phi[i]=atan(num[i]*.5/aa)*2;
          ph[i]=phi[i]*180/pi;
          bet[i]=asin((c*sin(phi[i])-a*sin(thet))/b);
          betd[i]=bet[i]*180/pi;
          velc[i]=(a*vela*sin(bet[i]-thet))/(c*sin(bet[i]
          -phi[i]));
          velb[i]=(a*vela*sin(phi[i]-thet))/(b*sin(bet[i]
          -phi[i]));
          c1[i]=a*acca*sin(bet[i]-thet);
          c2[i]=a*pow(vela,2)*cos(bet[i]-thet)+
          b*pow(velb[i],2);
          c3[i]=c*pow(velc[i],2)*cos(phi[i]-bet[i]);
          c4[i]=c*sin(bet[i]-phi[i]);
          accc[i]=(c1[i]-c2[i]+c3[i])/c4[i];
          b1[i]=a*acca*sin(phi[i]-thet);
          b2[i]=a*pow(vela,2)*cos(phi[i]-thet);
          b3[i]=b*pow(velb[i],2)*cos(phi[i]-bet[i])
          -c*pow(velc[i],2);
          b4[i]=b*sin(bet[i]-phi[i]);
          accb[i]=(b1[i]-b2[i]-b3[i])/b4[i];
          printf( "%6.2d %6.2f%8.2f %8.2f %8.2f %6.2f
          %6.2f %6.2f %6.2f\n",th,vela,acca,ph[i],betd[i],
          velc[i],velb[i],accc[i],accb[i]);
        }
    }
    vela=sqrt(vela*vela+2*acca*dthet);
  }
    getch();
}
```

Fig. 4.2

Figure 4.2 shows a program in *C* for solving such a problem. The program can be used to find the angular velocities and accelerations of the output and coupler links for the following cases:

1. Link *AB* is a crank and rotates at uniform angular velocity. In this case, the acceleration of the input link will be zero. If the link *AB* is not a crank but a rocker, the program will make the calculations only for feasible cases.
2. Link *AB* is a crank and starts from the stationary position. In this case, the initial velocity is zero and a value of the acceleration has to be provided along with the limit of the angle up to which the acceleration continues. At that angle when the maximum velocity is attained, the acceleration automatically reduces to zero and the onward the crank starts rotating at constant angular velocity. Further, calculations are made for one complete revolution.
3. For instant values of input velocity and acceleration, only one calculation is made for that specified position.

Various input variables are

`a, b, c, d`	Magnitudes of links *AB*, *BC*, *CD* and *DA* respectively (mm)
`vela`	Angular velocity of the input link *AB* (m/s)
`acca`	Angular acceleration of the input link (m/s^2)(acceleration is taken positive, deceleration negative)
`theta`	The interval of the input angle, i.e., the results are to be taken with a difference of 10°, 20° or 30°, etc., starting from zero
`Limit`	Angle up to which acceleration continues (for the case 2; in the other cases any value may be given)

The output variables are

`thet`	Angular displacement of the input link *AB* (degrees)
`phi`	Angular displacement of the output link *DC* (degrees)
`beta`	Angular displacement of the coupler link *BC* (degrees)
`velc`	Angular velocity of the output link *DC* (rad/s)
`velb`	Angular velocity of the coupler link *BC* (rad/s)
`accc`	Angular acceleration of the output link (rad/s^2)
`accb`	Angular acceleration of the coupler link (rad/s^2)

The results are obtained in sets of two possible solutions for each position of the input link. In case the input *AB* is not a crank, the results are obtained for the possible positions only. The counter-clockwise direction is considered as positive and the clockwise as negative.

4.2 USE OF COMPLEX ALGEBRA

For a four-link mechanism, we can write

$$\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d} = \mathbf{0} \tag{4.24}$$

Transforming it into complex polar form,

$$a\, e^{i\theta} + b\, e^{i\beta} - c\, e^{i\phi} - d = 0 \tag{4.25}$$

Now, we know, $e^{i\theta} = \cos\theta + i \sin\theta$

Thus, transforming this equation into complex rectangular form and separating the real and imaginary terms,

$$a \cos\theta + b \cos\beta = d + c \cos\phi \tag{4.26}$$

and

$$a \sin\theta + b \sin\beta = c \sin\phi \tag{4.27}$$

which are the same equations as 4.1 and 4.3 and thus can be solved to find β and θ.

Differentiating Eq. (4.25) with respect to t,

$$ia\dot{\theta}\,e^{i\theta} + ib\dot{\beta}\,e^{i\beta} - ic\dot{\varphi}\,e^{i\varphi} = 0 \tag{4.28}$$

or

$$ia\,\omega_a\,e^{i\theta} + ib\,\omega_b\,e^{i\beta} - ic\,\omega_c\,e^{i\varphi} = 0 \tag{4.29}$$

Again, transforming this equation into complex rectangular form and separating the real and imaginary terms,

$$a\,\omega_a \cos\theta + b\omega_b \cos\beta - c\,\omega_c \cos\varphi = 0 \tag{4.30}$$

$$-a\,\omega_a \sin\theta - b\omega_b \sin\beta + c\,\omega_c \sin\varphi = 0 \tag{4.31}$$

which are the same equations as 4.13 and 4.15 and thus can be solved to find ω_b and ω_c.

Differentiating Eq. (4.28) with respect to t,

$$ia(\ddot{\theta}e^{i\theta} + i\dot{\theta}^2 e^{i\theta}) + ib(\ddot{\beta}\;e^{i\beta} + i\dot{\beta}^2 e^{i\beta}) - ic(\ddot{\varphi}e^{i\varphi} + i\dot{\varphi}^2 e^{i\varphi}) = 0 \tag{4.32}$$

or

$$ia(\alpha_a e^{i\theta} + i\omega_a{}^2 e^{i\theta}) + ib(\alpha_b\ddot{\beta}\;e^{i\beta} + i\omega_b{}^2 e^{i\beta}) - ic(\alpha_c e^{i\varphi} + i\omega_c{}^2 e^{i\varphi}) = 0 \tag{4.33}$$

Transforming this equation into complex rectangular form and separating the real and imaginary terms,

$$-a\alpha_a \sin\theta - a\omega_a^2 \cos\theta - b\alpha_b \sin\beta - b\omega_b^2 \cos\beta + c\alpha_c \sin\varphi + c\omega_c^2 \cos\varphi = 0 \tag{4.34}$$

$$a\alpha_a \cos\theta - a\omega_a^2 \sin\theta + b\alpha_b \cos\beta - b\omega_b^2 \sin\beta - c\alpha_c \cos\varphi + c\omega_c^2 \sin\varphi = 0 \tag{4.35}$$

which are the same equations as 4.20 and 4.21 and can be solved as before.

4.3 THE VECTOR METHOD

We have

$$\mathbf{a} + \mathbf{b} - \mathbf{c} - \mathbf{d} = \mathbf{0}$$

Assuming that the angles β and ϕ have been determined by any of the above methods, differentiate the above equation with respect to time,

$$\omega_{\mathbf{a}} \times \mathbf{a} + \omega_{\mathbf{b}} \times \mathbf{b} - \omega_{\mathbf{c}} \times \mathbf{c} = \mathbf{0} \qquad (a, b, c \text{ and } d \text{ are constants}) \tag{4.36}$$

Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be the unit vectors along **a**, **b** and **c** vectors. In plane–motion mechanisms, all the angular velocities are in the **k** direction. Therefore,

$$a\omega_a\,(\mathbf{k} \times \hat{\mathbf{a}}) + b\,\omega_b\,(\mathbf{k} \times \hat{\mathbf{b}}) - c\,\omega_c\,(\mathbf{k} \times \hat{\mathbf{c}}) = \mathbf{0} \tag{4.37}$$

Take the dot product with $\hat{\mathbf{b}}$,

$$a\,\omega_a\,(\mathbf{k} \times \hat{\mathbf{a}}).\,\hat{\mathbf{b}} + b\omega_b\,(\mathbf{k} \times \hat{\mathbf{b}})\,\hat{\mathbf{b}} - c\,\omega_c\,(\mathbf{k} \times \hat{\mathbf{c}})\,\hat{\mathbf{b}} = 0$$

$$a\,\omega_a\,(\mathbf{k} \times \hat{\mathbf{a}}).\,\hat{\mathbf{b}} + \mathbf{0} - c\omega_c\,(\mathbf{k} \times \hat{\mathbf{c}})\,\hat{\mathbf{b}} = 0$$

or

$$\omega_c = -\frac{a\omega_a\,(\mathbf{k} \times \hat{\mathbf{a}}).\,\hat{\mathbf{b}}}{c(\mathbf{k} \times \hat{\mathbf{b}}).\,\hat{\mathbf{b}}} \tag{4.38}$$

Taking the dot product with $\hat{\mathbf{c}}$,

$$a\,\omega_a\,(\mathbf{k} \times \hat{\mathbf{a}}).\,\hat{\mathbf{c}} + b\,\omega_b\,(\mathbf{k} \times \hat{\mathbf{b}}).\,\hat{\mathbf{c}} - c\,\omega_c\,(\mathbf{k} \times \hat{\mathbf{c}})\,\hat{\mathbf{c}} = 0$$

$$a\,\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{c}} + b\,\omega_b\,(\mathbf{k}\times\hat{\mathbf{b}})\,.\,\hat{\mathbf{c}} - 0 = 0$$

or

$$\omega_b = -\frac{a\,\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{c}}}{b(\mathbf{k}\times\hat{\mathbf{b}})\,.\,\hat{\mathbf{c}}} \qquad (4.39)$$

It can be shown that Eqs 4.38 and 4.39 are the same as Eqs 4.16 and 4.17 as follows:

$$(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix}\,.\,\hat{\mathbf{b}} = (-\sin\theta.\mathbf{i} + \cos\theta.\mathbf{j})\,.\,\hat{\mathbf{b}}$$

$$= (-\sin\theta.\mathbf{i} + \cos\theta.\mathbf{j})\,.\,(\cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j})$$

$$= -\sin\theta\cos\beta + \cos\theta\sin\beta$$

$$= \sin(\beta - \theta)$$

Similarly,

$$(\mathbf{k}\times\hat{\mathbf{c}})\,.\,\hat{\mathbf{b}} = \sin(\beta - \phi)$$

Therefore,

$$\omega_c = -\frac{a\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}}).\hat{\mathbf{b}}}{c(\mathbf{k}\times\hat{\mathbf{c}}).\hat{\mathbf{b}}} = \frac{a\omega_a\sin(\beta-\theta)}{c\sin(\beta-\varphi)} \qquad (4.40)$$

In the same way,

$$w_b = -\frac{a\,\omega_a\,(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{c}}}{b(\mathbf{k}\times\hat{\mathbf{c}})\,.\,\hat{\mathbf{c}}} = -\frac{a\omega_a\sin(\varphi-\theta)}{b\sin(\varphi-\beta)} \qquad (4.41)$$

which are the same equations as equations 4.16 and 4.17.

Differentiating Eq. 4.36 with respect to time to get the accelerations,

$$\dot{\omega}_a\times\mathbf{a} + \omega_a\times(\omega_a\times\mathbf{a}) + \dot{\omega}_b\times\mathbf{b} + \omega_b\times(\omega_b\times\mathbf{b}) - \dot{\omega}_c\times\mathbf{c} - \omega_c\times(\omega_c\times\mathbf{c}) = 0$$

or

$$\alpha_a\times\mathbf{a} + \omega_a\times(\omega_a\times\mathbf{a}) + \alpha_b\times\mathbf{b} + \omega_b\times(\omega_b\times\mathbf{b}) - \alpha_c\times\mathbf{c} - \omega_c\times(\omega_c\times\mathbf{c}) = 0$$

or

$$a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}}) - a\,\omega_a^2\,\hat{\mathbf{a}} + b\,\alpha_b(\mathbf{k}\times\hat{\mathbf{b}}) - b\,\omega_b^2\,\hat{\mathbf{b}} - c\alpha_c(\mathbf{k}\times\hat{\mathbf{c}}) + c\,\omega_c^2\,\hat{\mathbf{c}} = 0 \qquad (4.42)$$

Take the dot product of this equation with $\hat{\mathbf{b}}$,

$$a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{b}} - a\,\omega_a^2\,\hat{\mathbf{a}}\,.\,\hat{\mathbf{b}} + 0 - b\,\omega_b^2\,\hat{\mathbf{b}}\,.\,\hat{\mathbf{b}} - c\alpha_c(\mathbf{k}\times\hat{\mathbf{c}})\,.\,\hat{\mathbf{b}} + c\,\omega_c^2\,\hat{\mathbf{c}}\,.\,\hat{\mathbf{b}} = 0$$

$$\alpha_c = \frac{a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{b}} - a\,\omega_a^2\,\hat{\mathbf{a}}\,.\,\hat{\mathbf{b}} - b\omega_b^2 + c\omega_c^2\,\hat{\mathbf{c}}\,.\,\hat{\mathbf{b}}}{c(\mathbf{k}\times\hat{\mathbf{c}})\,.\,\hat{\mathbf{b}}} \qquad (4.43)$$

Since,

$$(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{b}} = \sin(\beta-\theta),$$
$$\hat{\mathbf{a}}\,.\,\hat{\mathbf{b}} = \cos(\beta-\theta),$$
$$\hat{\mathbf{b}}\,.\,\hat{\mathbf{b}} = t$$
$$\hat{\mathbf{c}}\,.\,\hat{\mathbf{b}} = \cos(\beta-\varphi)$$
$$(\mathbf{k}\times\hat{\mathbf{c}})\,.\,\hat{\mathbf{b}} = \text{win}\,(\beta-\varphi)$$

The above equation reduces to

$$\alpha_c = \frac{a\alpha_a\sin(\beta-\theta) - a\omega_a^2\cos(\beta-\theta) - b\omega_b^2 + c\omega_c^2\cos(\beta-\varphi)}{c\sin(\beta-\varphi)} \tag{4.44}$$

which is the same as Eq. 4.23.

Taking the dot product of Eq. 4.42 with $\hat{\mathbf{c}}$,

$$a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{b}} - a\,\omega_a^2\,\hat{\mathbf{a}}\,.\,\hat{\mathbf{b}} + b\,\alpha_b(\mathbf{k}\times\hat{\mathbf{b}})\,.\,\hat{\mathbf{c}} - b\,\omega_b^2\,\hat{\mathbf{b}}\,.\,\hat{\mathbf{c}} + c\,\omega_c^2\,\hat{\mathbf{c}}\,.\,\hat{\mathbf{c}} = 0$$

or

$$b = \frac{a\,\alpha_a(\mathbf{k}\times\hat{\mathbf{a}})\,.\,\hat{\mathbf{c}} - a\,\omega_a^2\,\hat{\mathbf{a}}\,.\,\hat{\mathbf{c}} - b\omega_b^2\,\hat{\mathbf{b}}\,.\,\hat{\mathbf{c}} + c\omega_c^2}{c(\mathbf{k}\times\hat{\mathbf{c}})\,.\,\hat{\mathbf{b}}} \tag{4.45}$$

which can be shown to be the same as Eq. 4.22, i.e.,

$$\alpha_b = \frac{a\alpha_a\sin(\varphi-\theta) - a\omega_a^2\cos(\varphi-\theta) - b\omega_b^2\cos(\varphi-\beta) + c\omega_c^2}{b\sin(\beta-\varphi)}$$

Example 4.1 *In a four-link mechanism, the dimensions of the links are as under:*
AB = 50 mm, BC = 66 mm, CD = 56 mm and AD = 100 mm
AD is the fixed link. At an instant when DAC is 60°, the angular velocity of the input link AB is 10.5 rad/s in the counter-clockwise direction with an angular retardation of 26 rad/s². Determine analytically the angular displacements, angular velocities and angular accelerations of the output link DC and the coupler BC.

Solution We have,

$2k = a^2 - b^2 + c^2 + d^2$

$k = (50^2 - 66^2 + 56^2 + 100^2)/2$

$= 5640$

$A = k - a\,(d - c)\cos\theta - cd$

$= 5640 - 50\,(100 - 56)\cos 60° - 56\times 100 = -1060$

$B = -2ac\sin\theta = -2\times 50\times 56\sin 60° = -4850$

$C = k - a\,(d + c)\cos\theta + cd$

$= 5640 - 50\,(100 + 56)\cos 60° + 56\times 100 = 7340$

$$\varphi = 2\tan^{-1}\left[\frac{-B\pm\sqrt{B^2-4AC}}{2A}\right]$$

$$= 2\tan^{-1}\left[\frac{4850\pm\sqrt{(-4850)^2-4\times(-1060)(7340)}}{2\times(-1060)}\right]$$

$= 2\tan^{-1}(1.199 \text{ or } -5.759)$

$= 100.35° \text{ or } -160.3°$

Taking the first value,

we have,

$b\sin\beta = c\sin\varphi - a\sin\theta$

$66\times\sin\beta = 56\times\sin 100.35° - 50\times\sin 60°$

$\sin\beta = 0.1786$

$\beta = 10.29°$

$$\omega_c = \frac{a\omega_a\sin(\beta-\theta)}{c\sin(\beta-\varphi)}$$

$$= \frac{50 \times 10.5 \sin(10.29 - 60^\circ)}{56 \sin(10.29 - 100.35)} = 7.15 \text{ rad/s}$$

$$\omega_b = -\frac{a\omega_a \sin(\varphi - \theta)}{b \sin(\varphi - \beta)} =$$

$$-\frac{50 \times 10.5 \sin(100.35^\circ - 60^\circ)}{66 \times \sin(100.35^\circ - 10.29^\circ)} = -5.15 \text{ rad/s}$$

$$\alpha_c = \frac{a\alpha_a \sin(\beta - \theta) - a\omega_a^2 \cos(\beta - \theta) - b\omega_b^2 + c\omega_c^2 \cos(\beta - \varphi)}{c \sin(\beta - \varphi)}$$

$$= \frac{50 \times (-26)\sin(10.29^\circ - 60^\circ) - 50 \times 10.5^2 \cos(10.29^\circ - 60^\circ) - 66 \times (5.15)^2 + 56^2 \cos(10.29^\circ - 100.35^\circ)}{56 \sin(10.29^\circ - 100.35^\circ)}$$

$$= 77.26 \text{ rad/s}^2$$

$$\alpha_b = \frac{a\alpha_a \sin(\varphi - \theta) - a\omega_a^2 \cos(\varphi - \theta) - b\omega_b^2 \cos(\varphi - \beta) + c\omega_c^2}{b \sin(\beta - \varphi)}$$

$$= \frac{50 \times (-26)\sin(100.35^\circ - 60^\circ) - 50 \times 10.5^2 \cos(100.35^\circ - 60^\circ) - 66 \times (5.15)^2 \cos(100.35^\circ - 10.29^\circ) + 56 \times 7.15^2}{56 \sin(10.29^\circ - 100.35^\circ)}$$

$$= 32.98 \text{ rad/s}^2$$

Using the other value of ϕ, ($\phi = -160.3^\circ$), another set of values of velocities and accelerations can be obtained.

The results obtained using the program of Fig. 4.2 are given in Fig. 4.3.

```
Enter values of a, b, c, d, vela, acca, theta, limit
50  66  56  100  10.5  -26  60  0
thet    vela     acca      phi      beta      velc     velb     accc     accb
60      10.5     -26.00   -160.35  -70.29    -7.15     5.15    50.04    94.32
60      10.50    -26.00    100.35   10.29     7.15    -5.15    77.26    32.98
```

Fig. 4.3

[Compare these values of ω_b, ω_c, α_b and α_c at 60° with the values obtained graphically in Examples 2.1 and 3.1]

Example 4.2 *In a four-link mechanism, the dimensions of the links are as under:*
AB = 20 mm, BC = 66 mm, CD = 56 mm and AD = 80 mm
AD is the fixed link. The crank AB rotates at uniform angular velocity of 10.5 rad/s in the counter-clockwise direction. Determine using the program of Fig. 4.2, the angular displacements, angular velocities and angular accelerations of the output link DC and the coupler BC for a complete revolution of the crank at an interval of 40°.

Solution The results obtained using the program of Fig. 4.2 are given in Fig. 4.4.

```
Enter values of a, b, c, d, vela, acca, theta, limit
20  66  56  80  10.5  40  0
thet    vela     acca     phi      beta      velc     velb     aqccc    accb
00      10.5     0.0    -110.74   -52.51    -3.50    -3.50    -37.58    18.56
00      10.5     0.0     110.74    52.51    -3.50    -3.50     37.58   -18.56
40      10.5     0.0    -126.30   -61.47    -4.06    -0.83     15.50    50.99
40      10.5     0.0     103.82    38.99     0.07    -3.15     56.46    20.96
80      10.5     0.0    -139.02   -58.74    -2.51     2.03     26.17    31.04
80      10.5     0.0     110.16    29.87     2.92    -1.62     27.30    22.42
```

(contd.)

120	10.5	0.0	-145.28	-48.22	-0.77	3.20	26.64	4.54
120	10.5	0.0	123.49	26.44	3.77	-0.20	-0.66	21.44
160	10.5	0.0	-144.69	-36.44	1.12	2.75	29.94	-16.40
160	10.5	0.0	136.77	28.52	2.96	1.32	-22.41	23.99
200	10.5	0.0	-136.77	-28.52	2.96	1.32	22.41	-23.99
200	10.5	0.0	144.69	36.44	1.12	2.75	-29.94	16.40
240	10.5	0.0	-123.49	-26.44	3.77	-0.20	0.66	-21.44
240	10.5	0.0	145.28	48.22	-0.77	3.20	-26.64	-4.54
280	10.5	0.0	-110.16	-29.87	2.92	-1.62	-27.30	22.42
280	10.5	0.0	139.02	58.74	2.51	2.03	-26.17	-31.04
320	10.5	0.0	-103.82	-38.99	0.07	-3.15	-56.46	-20.96
320	10.5	0.0	126.30	61.47	-4.06	-0.83	-15.50	-50.99

Fig. 4.4

4.4 SLIDER-CRANK MECHANISM

Figure 4.5 shows a slider-crank mechanism in which the strokeline of the slider does not pass through the axis of rotation of the crank. Angle β in clockwise direction from the x-axis is taken as negative.

Let e = eccentricity (distance CD).

Displacement along x-axis,

$$a \cos\theta + b \cos(-\beta) = d \tag{4.46}$$

or

$$b \cos\beta = d - a \cos\theta \tag{4.46a}$$

Displacement along y-axis,

$$a \sin\theta + b \sin(-\beta) + e \tag{4.47}$$

or

$$b \sin\beta = e - a \sin\theta \tag{4.47a}$$

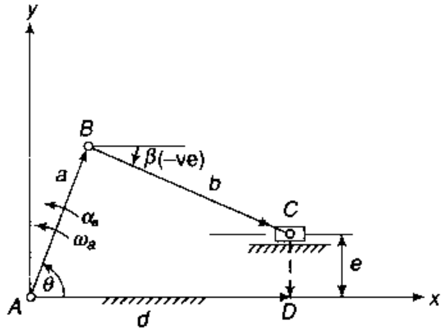


Fig. 4.5

Squaring Eqs (4.46a) and (4.47a) and adding,

$$b^2 = a^2 \cos^2\theta + d^2 - 2ad \cos\theta + a^2 \sin^2\theta + e^2 - 2ae \sin\theta$$

$$= a^2 + e^2 + d^2 - 2ae \sin\theta - 2ad \cos\theta$$

or

$$d^2 - (2a \cos\theta)d + a^2 - b^2 + e^2 - 2ae \sin\theta = 0$$

or

$$d^2 + C_1 d + C_2 = 0 \tag{4.48}$$

where $C_1 = -2a \cos\theta$

$C_2 = a^2 - b^2 + e^2 - 2ae \sin\theta$

Equation (4.48) is a quadric in d. Its two roots are,

$$d = \frac{-C_1 \pm \sqrt{C_1^2 - 4C_2}}{2} \quad (4.49)$$

Thus, if the parameters a, b, e and θ of the mechanism are known, the output displacement can be computed.

Also, from Eq. (4.47a),

$$\beta = \sin^{-1} \frac{e - a \text{in}\,\theta}{b} \quad (4.50)$$

Velocity Analysis

Differentiating Eqs. (4.46) and (4.47) with respect to time,

$$-a\omega_a \sin\theta - b\,\omega_b \sin\beta - \dot{d} = 0 \quad (4.51)$$

$$a\omega_a \cos\theta + b\,\omega_b \cos\beta = 0 \quad (4.52)$$

Multiply Eq. (4.51) by $\cos\beta$ and Eq. (4.52) by $\sin\beta$ and add,

$$a\omega_a (\sin\beta \cos\theta - \cos\beta \sin\theta) - \dot{d}\cos\beta = 0$$

$$\dot{d} = \frac{a\omega_a \sin(\beta - \theta)}{\cos\beta} = \quad (4.53)$$

From Eq. (4.52),

$$\omega_b = -\frac{a\omega_a \cos\theta}{b\cos\beta} \quad (4.54)$$

ω_b provides the angular velocity of the coupler-link whereas $\dot{d}$ gives the linear velocity of the slider.

Acceleration Analysis

Differentiating Eqs (4.51) and (4.52) with respect to time,

$$-\left[a\alpha_a \sin\theta + a\omega_a^2 \cos\theta\right] - \left[b\alpha_b \sin\beta + b\omega_b^2 \cos\beta\right] - \ddot{d} = 0 \quad (4.55)$$

$$\left[a\alpha_a \cos\theta + a\omega_a^2 \sin\theta\right] - \left[b\alpha_b \cos\beta + b\omega_b^2 \sin\beta\right] = 0 \quad (4.56)$$

Multiply Eq. (4.55) by $\cos\beta$ and Eq. (4.56) by $\sin\beta$ and add,

or $$a\alpha_a \sin(\beta - \theta) - a\omega_a^2 \cos(\beta - \theta) - b\omega_b^2 - \ddot{d}\cos\beta = 0$$

$$\ddot{d} = \frac{a\alpha_a \sin(\beta - \theta) - a\omega_a^2 \cos(\beta - \theta) - b\omega_b^2}{\cos\beta} \quad (4.57)$$

From Eq. (4.56)

$$\alpha_b = \frac{a\alpha_a \cos\theta - a\omega_a^2 \sin\theta - b\omega_b^2 \sin\beta}{b\cos\beta} \quad (4.58)$$

α_b provides the angular acceleration of the coupler-link whereas $\ddot{d}$ gives the linear acceleration of the slider.

Figure 4.6 shows a program to solve this type of problem. It can be used for the same type of three cases as for the four-link mechanism.

```
#include<stdio.h>
#include<conio.h>
#inculde<math.h>
void main()
{
    int j,iht,th,theta,limit,ins;
    float a,b,e,c1,c2,c3,c4,vela,acca,thet,pi,dthet,bet,
    velb,vels,accs,accb;
    clrscr();
    printf("enter values a,b,e,vela,acca,theta,limit\n");
    scanf("%f%f%f%f%f%d%d%",&a,&b,&e,&vela,&acca,&theta,
    &li    mit);
    printf( " thet vela acca beta ");
    printf( " velc velb accc accb \n");
    ins=1;
    if(vela==0 && acca>0)ins=0;
    pi=4*atan(1);
    iht=360/theta;
    if(vela>0 && acca==00) {ins=0;iht=360/theta; }
    if(ins==1)iht=theta;
    dthet=pi*2/iht;
    if(vela==0 && acca>0)iht=iht+limit/theta;
          for(j=0;j<iht+1;j++)
          {
            if(j>(iht-360/theta-1) && ins==0)acca=0;
            thet=j*dthet;
            if(ins==1) {j=iht; thet=theta*pi/180;}
            th=theta*j;
            if(ins==1)th=theta;
            bet=asin((e-a*sin(thet))/b);
            vels=-a*vela*sin(thet-bet)/(cos(bet)*1000);
            velb=-a*vela*cos(thet)/b*cos(bet);
            c1=a*acca*sin(bet-thet)-b*pow(velb,2);
            c2=a*pow(vela,2)*cos(bet-thet);
            accs=(c1-c2)/(cos(bet)*1000);
            c3=a*acca*cos(thet)-a*pow(vela,2)*sin(thet);
            c4=b*pow(velb,2)*sin(bet);
            accb=-(c3-c4)/(b*cos(bet));
            printf( "%6.2d %6.2f %6.2f %6.2f %6.2f%8.2f
            %8.2f %8.2f\n",th,vela,acca,bet*180/pi,vels,
            velb,accs,accb);
            vela=sqrt(vela*vela+2*acca*dthet);

          }
      getch();
  }
```

Fig. 4.6

The input variables are

`a,b,e`	The magnitudes *a*, *b* and *e* respectively (mm)
`vela`	Angular velocity of the input link *AB* (m/s)
`acca`	Angular acceleration of the input link (m/s^2)
`theta`	The interval of the input angle (degrees)
`limit`	Angle up to which acceleration continues, in case the crank starts from stationary position (in other cases any value may be given)

The output variables are

`thet`	Angular displacement of the input link *AB* (degrees)
`bet`	Angular displacement of link *AB* (rad/s)
`vels`	Linear velocity of the slider (m/s)
`velb`	Angular velocity of link *BC* (rad/s)
`accs`	Linear acceleration of the slider (m/s^2)
`accb`	Angular acceleration of link *BC* (rad/s^2)

Example 4.3 ***In a slider-crank mechanism, the lengths of the crank and the connecting rod are 480 mm and 1.6 m respectively. It has an eccentricity of 100 mm. Assuming a velocity of 20 rad/s of the crank OA, calculate the following at an interval of 30°:***

(i) Velocity and the acceleration of the slider

(ii) Angular velocity and angular acceleration of the connecting rod

Solution The input and the output have been shown in Fig. 4.7. The results have been obtained at an interval of 30° of the input link (crank).

```
Enter values of a, b, e, vela, acca, theta, limit
480  1600  100  20  0  30  0
```

thet	vela	acca	beta	vels	velb	accs	accb
00	20.0	0.0	3.58	0.60	-5.99	-249.49	2.25
30	20.0	0.0	-5.02	-5.53	-5.18	-200.88	57.88
60	20.0	0.0	-11.38	-9.28	-2.94	-76.65	104.27
90	20.0	0.0	-13.74	-9.60	0.00	46.94	123.53
120	20.0	0.0	-11.38	-7.35	2.94	115.35	104.27
150	20.0	0.0	-5.02	-4.07	5.18	131.67	57.88
180	20.0	0.0	3.58	-0.60	5.99	134.51	2.25
210	20.0	0.0	12.27	2.99	5.08	144.94	-55.80
240	20.0	0.0	18.80	6.68	2.84	138.98	-107.04
270	20.0	0.0	21.25	9.60	-0.00	74.68	-128.76
300	20.0	0.0	18.80	9.95	-2.84	-53.02	-107.04
330	20.0	0.0	12.27	6.61	-5.08	-187.61	-55.80

Fig. 4.7

COUPLER CURVES

A coupler curve is the locus of a point on the coupler link. A four-link mechanism *ABCD* with a coupler point *E* (offset) is shown in Fig. 4.8. Let the *x*-axis be along the fixed link *AD*.

Let $BE = e$ and $\angle CBE = \alpha$

Angles β and γ are defined as shown in the diagram.

Let X_e and Y_e be the coordinates of the point *E*.

Then,

$$X_e = a\cos\theta + e\cos(\alpha+\beta) \tag{4.59}$$

$$Y_e = a\sin\theta + e\sin(\alpha+\beta) \tag{4.60}$$

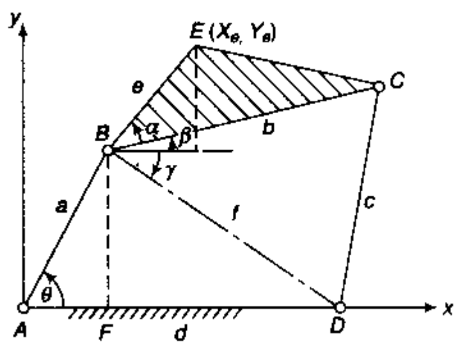


Fig. 4.8

In these equations a, e, θ and α are known. To know the coordinates X_e and Y_e, it is necessary to express β in terms of known parameters, i.e., a, b, c, d, e, θ and α.

In ΔBDC, applying cosine law,

$$\cos(\beta+\gamma) = \frac{b^2+f^2-c^2}{2bf}$$

or

$$\beta+\gamma = \cos^{-1}\left[\frac{b^2+f^2-c^2}{2bf}\right]$$

$$\beta = \cos^{-1}\left[\frac{b^2+f^2-c^2}{2bf}\right] - \gamma \tag{4.61}$$

where

$$\tan\gamma = \frac{BF}{FD} = \frac{BF}{AD-AF} = \frac{a\sin\theta}{d-a\cos\theta}$$

or

$$\gamma = \tan^{-1}\left[\frac{a\sin\theta}{d-a\cos\theta}\right] \tag{4.62}$$

f^2 can be found by applying the cosine law to ΔABD,

i.e.,

$$f^2 = a^2 + d^2 - 2ad\cos\theta$$

Having found the value of the angle β, the coordinates of the point *E* can be known for different values of θ from Eqs (4.59) and (4.60).

A coupler curve can also be obtained in case of a slider-crank mechanism (Fig. 4.9). The angle *CBE* is α and the eccentricity is e.

Draw $BL \perp AD$ and $CF \perp BL$

$$X_e = a\cos\theta + e\cos(\alpha-\beta) \tag{4.63}$$

$$Y_e = a\sin\theta + e\sin(\alpha-\beta) \tag{4.64}$$

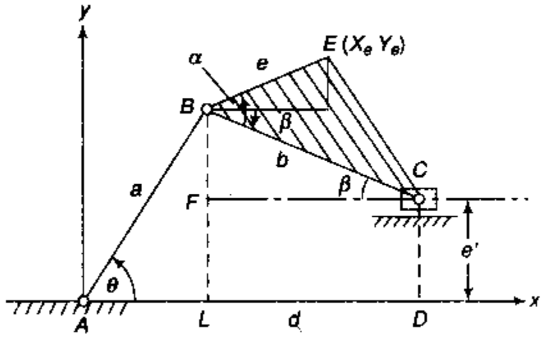


Fig. 4.9

β (negative) can be expressed in terms of known parameters as below:

$$\sin\beta = \frac{BF}{BC} = \frac{BL - FL}{BC} = \frac{a\sin\theta - e'}{b}$$

$$\beta = \sin^{-1}\left[\frac{a\sin\theta - e'}{b}\right] \tag{4.65}$$

Figure 4.10 shows a program to find the coordinates of the coupler point for both the above cases. The input variables are

`a, b, e`	The magnitudes a, b and e respectively (mm)
`case`	1, in case of a four-link mechanism 2, in case of a slider-crank mechanism
`c`	The magnitude c (case 1) or eccentricity e' (case 2)
`d`	The magnitude d (case 1) or 0 (case 2)
`alph`	The angle α (degrees)

The output variables are:

`thet`	Angular displacement of the link AB (degrees)
`xe`	X-coordinates of the point E
`ye`	Y-coordinates of the point E

```
#inculde<stdio.h>
#inculde<conio.h>
#inculde<math.h>
void main()
{
    int cas;
    float a,b,c,d,e,f,alph,gamm,bet,squ,pi,thet,theta,
    xe,ye;
    clrscr();

    printf("enter values of a,b,c,d,e,alph,cas,theta\n");
    scaf("%f%f%f%f%f%f%d%f",&a,&b,&c,&d,&e,&alph,
    &cas,theta);
    printf("     theta          xe          ye\n");
    thet=0;
    pi=4*atan(1);
        while(thet<359*pi/180)
        {
            gamm=atan(a*sin(thet)/(d-a*cos(thet)));
            squ=a*a+d*d-2*a*d*cos(thet);
            f=pow(squ,.5);
            if (cas==1)bet=acos((b*b+f*f-c*c)/2*b*f))-gamm;
            if (cas==2) bet=asin((c-a*sin(thet))/b);
            xe=a*cos(thet)+e*cos(alph*pi/180+bet);
            ye=a*sin(thet)+e*sin(alph*pi/180+bet);
            printf( "%10.2f %10.2f %10.2f \n",
            thet*180/pi,xe,ye);thet=thet+theta*pi/180;
        }
    getch();
}
```

Fig. 4.10

Example 4.4 *Draw a coupler curve of the coupler point E of a four-link mechanism having the following data:*

AB = 50 mm, BC = 66 mm, CD = 90 mm,

AD = 100 mm, BE = 30 mm ∠CBE = 40°
(refer to Fig. 4.8)

```
Enter values of a, b, c, d, e, alph,
cas theta
50  66  90  100  30  40  1  30
     theta          xe          ye
      0.0        26.73       18.93
     30.0        35.27       53.90
     60.0        29.79       72.92
     90.0        11.70       77.62
    120.0        -9.51       68.99
    150.0       -26.10       49.58
    180.0       -34.41       25.63
    210.0       -35.43        3.95
    240.0       -28.72      -13.53
    270.0       -15.08      -24.06
    300.0         1.75      -24.35
    330.0        16.54      -11.44
```

Fig. 4.11

Solution The input and the output have been shown in Fig. 4.11 using the program of Fig. 4.10 for the given date. The required coupler curve has been shown in Fig. 4.12.

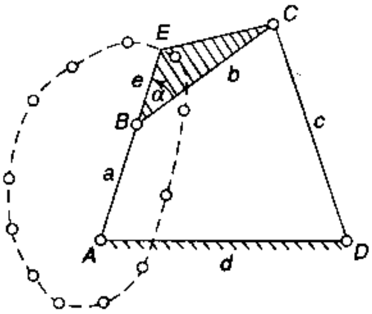


Fig. 4.12

Summary

1. To draw velocity and acceleration diagrams again and again for different positions of the crank is not convenient. Analytical methods prove to be very helpful.
2. In analytical methods, the links of the mechanism are considered as vectors.
3. In a four-link mechanism,
   (i) The angle of the output link is given by

$$\varphi = 2\tan^{-1}\left[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right]$$

where
$2k = a^2 - b^2 + c^2 + d^2$
$A = k - a(d - c)\cos\theta - cd$
$B = -2ac\sin\theta$
$C = k - a(d + c)\cos\theta + cd$

   (ii) The angle of the coupler link is given by

$$\beta = 2\tan^{-1}\left[\frac{-E \pm \sqrt{E^2 - 4DF}}{2D}\right]$$

where
$2k' = a^2 + b^2 - c^2 + d^2$
$D = k' - a(d + b)\cos\theta + bd$
$E = 2ab\sin\theta$
$F = k' - a(d - b)\cos\theta - bd$

   (iii) The velocities of the output and coupler links are given by

$$\omega_c = \frac{a\omega_a \sin(\beta - \theta)}{c\sin(\beta - \varphi)} \text{ and } \omega_b = -\frac{a\omega_a \sin(\varphi - \theta)}{b\sin(\varphi - \beta)}$$

   (iv) The accelerations of the output and coupler links are given by

$$a_c = \frac{a\alpha_a \sin(\beta - \theta) - a\omega_a^2\cos(\beta - \theta) - b\omega_b^2 + c\omega_c^2\cos(\beta - \varphi)}{c\sin(\beta - \varphi)}$$

and

$$a_b = \frac{a\alpha_a \sin(\varphi - \theta) - a\omega_a^2\cos(\varphi - \theta) - b\omega_b^2\cos(\varphi - \beta) + c\omega_c^2}{b\sin(\beta - \varphi)}$$

4. In a slider-crank mechanism,
   (i) The displacement of the slider is given by

$$d = \frac{-C_1 \pm \sqrt{C_1^2 - 4C_2}}{2}$$

where $C_1 = -2a\cos\theta$

$C_2 = a^2 - b^2 + e^2 - 2ae\sin\theta$

(ii) The angle of the coupler, $\beta = \sin^{-1}\dfrac{e - a\sin\theta}{b}$

(iii) The velocities of the slider and the angular velocity of the coupler are given by

$$\dot{d} = \frac{a\omega_a \sin(\beta-\theta)}{\cos\beta} \quad \text{and} \quad \omega_b = \frac{a\omega_a\cos\theta}{b\cos\beta}$$

(iv) The accelerations of the slider and the angular velocity of the coupler are given by

$$\ddot{d} = \frac{a\alpha_a\sin(\beta-\theta) - a\omega_a^2\cos(\beta-\theta) - b\omega_b^2}{\cos\beta} \quad \text{and}$$

$$\alpha_b = \frac{a\alpha_a\cos\theta - a\omega_a^2\sin\theta - b\omega_b^2\sin\beta}{b\cos\beta}$$

Exercises

1. Find expressions to determine the angles of the output link and coupler of a four-link mechanism. Deduce relations for the angular velocity and accelerations of the same links.
2. Deduce expressions to find the linear velocity and acceleration and angular velocity and angular acceleration of the coupler of a slider-crank mechanism.
3. What are coupler curves? Deduce expressions to draw the same in case of a four-link mechanism and slider-crank mechanism.
4. Derive expressions for the displacement, velocity and acceleration analyses of an inverted slider-crank mechanism.
5. In a four-link mechanism (Fig. 4.1), the dimensions of the links are AB = 30 mm, BC = 80 mm, CD = 40 mm and AD = 75 mm. If OA rotates at a constant angular velocity of 30 rad/s in the clockwise direction, calculate the angular velocities and the angular accelerations of links BC and CD for values of θ at an interval of 30°.
6. In a slider-crank mechanism (Fig. 4.5), the crank AB = 50 mm, BC = 160 mm and eccentricity e = 15 mm. For the angle θ = 45°, angular velocity of AB = 8 rad/s with an angular acceleration of 12 rad/s² (both clockwise), find the linear velocity and the acceleration of the slider and the angular velocity and the angular acceleration of the connecting rod analytically.

   (0.32 m/s, 1.98 m/s², 1.78 rad/s, 16.53 rad/s²)
7. Derive expressions to find the angular displacement, angular velocity and the angular acceleration of the link EF of a six-link mechanism shown in Fig. 4.13. AB is the input link having an angular velocity of ω rad/s in the counter-clockwise direction.

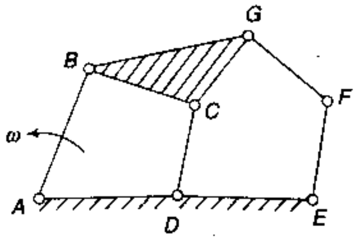


Fig. 4.13

8. Derive expressions for the coupler curves of an inverted slider-crank mechanism.
9. For the data of Example 4.3, take some more coupler points by taking different values of BE and $\angle\alpha$ and draw coupler curves for the same. Make a cardboard model of the mechanism and obtain the coupler curve by rotating the crank through 360°.

5 GRAPHICAL AND COMPUTER-AIDED SYNTHESIS OF MECHANISMS

Introduction

Dimensional synthesis of a pre-conceived type mechanism necessitates determining the principal dimensions of various links that satisfy the requirements of motion of the mechanism. A mechanism of preconceived type may be a four-link or a slider-crank mechanism. Principal dimensions involve link lengths, angular positions, position of pivots, eccentricities, angle between bell-crank levers and linear distance of sliders, etc. Synthesis of mechanisms may be done by graphical methods or by analytical means that involves the use of calculators and computers. In general, the types of synthesis may be classified as under:

1. *Function generation* It requires correlating the rotary or the sliding motion of the input and the output links. The motion of the output and the input links may be prescribed by an arbitrary function $y = f(x)$. This means if the input link moves by x, the output link moves by $y = f(x)$ for the range $x_o \le x \le x_{n+1}$ There lies n values of independent parameters $(x_1, x_2, \ldots .x_n)$ in the range between x_o and $x_{n=1}$. In case of rotary motions of the input and the output links, when the input link rotates through an angle θ, the output link moves through an angle ϕ corresponding to the value of the dependent variable $y = f(x)$. In case of slider-crank mechanism, the output is in the form of displacement s of the slider. It is to be noted that a four-link mechanism can match the function at only a limited number of prescribed points. However, it is a widely used mechanism in the industry since a four-bar is easy to construct and maintain and in most of the cases exact precision at many points is not required.
2. *Path generation* When a point on the coupler or the floating link of a mechanism is to be guided along a prescribed path, it is said to be a path generation problem. This guidance of the path of the point may or may not be coordinated with the movement of the input link and is generally called *with prescribed timing* or *without prescribed timing.*
3. *Motion generation* In this type, a mechanism is designed to guide a rigid body in a prescribed path. This rigid body is considered to be the coupler or the floating link of a mechanism.

If the above tasks are to be accomplished at fewer positions, it is simple to design a mechanism. However, when it is required to synthesize a mechanism to satisfy the input and the output links at larger number of positions, only an approximated solution can be obtained giving least deviation from the specified positions. In this chapter, both graphical as well as analytical methods to design a four-link mechanism and a slider-crank mechanism are being discussed.

PART A: GRAPHICAL METHODS

5.1 POLE

If it is desired to guide a body or link in a mechanism from one position to another, the task can easily be accomplished by simple rotation of the body about a point known as the *pole*. In Fig. 5.1, a link B_1C_1 is shown to move to another position B_2C_2 by rotating it about the pole P_{12}. This pole is easily found graphically by joining the midnormals of any two corresponding points on the link such as B_1B_2 and C_1C_2. If the pole point happens to fall off the frame of the machine, two fixed pivots, one each anywhere along the two midnormals will serve the purpose. In the figure, A and B are taken to be the fixed pivots. The configuration also happens to be a four-link mechanism $ABCD$ in two positions AB_1C_1D and AB_2C_2D in which the coupler link BC has moved from the position B_1C_1 to B_2C_2. The input link AB and the output link DC have moved through angles θ_{12} and ϕ_{12} respectively in the clockwise direction (Fig. 5.1).

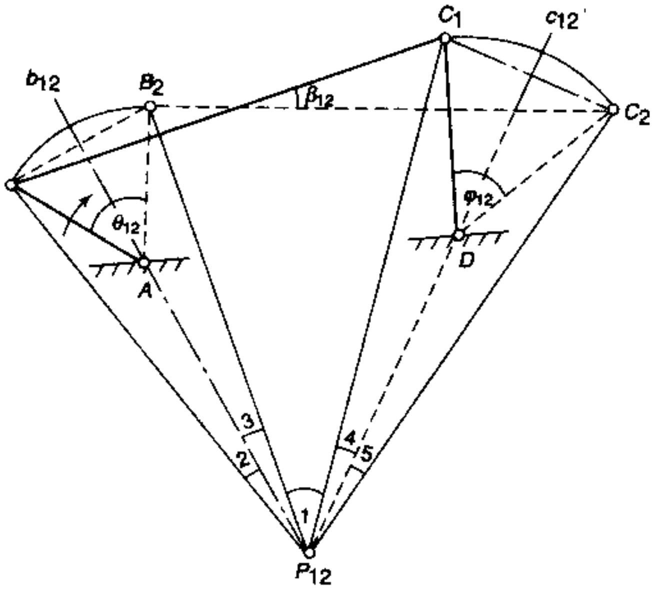


Fig. 5.1

Thus, a pole P_{12} of the coupler link BC is its centre of rotation with respect to the fixed link for the motion of the coupler from B_1C_1 to B_2C_2. Each point on the link BC describes a circular arc with centre at the pole P_{12}. Thus, a line joining the two positions of a point on the link is a chord of the circle and the midnormal (perpendicular bisector) of the chord passes through the centre of rotation P_{12}. B and C are also two points on the link BC. B moves from B_1 to B_2 while C from C_1 to C_2. Therefore, B_1B_2 and C_1C_2 are the chords of the two circles and their midnormals b_{12} and c_{12} also pass through or intersect at the centre of their rotation, i.e., at P_{12}.

Properties of Pole Point

1. As $AB_1 = AB_2$, the midnormal b_{12} of B_1B_2 passes through the fixed pivot A. Similarly, the midnormal of C_1C_2 passes through pivot D.
2. The coupler link BC is rotated about P_{12} from the position B_1C_1 to B_2C_2,

   $\therefore \quad \Delta B_1P_{12}C_1 \equiv \Delta B_2P_{12}C_2$

   $\therefore \quad \angle 2 + \angle 3 + \angle 1 = \angle 1 + \angle 4 + \angle 5$

   i.e., angle subtended by B_1C_1 at P_{12} = angle subtended by B_2C_2 at P_{12}
   or the angle subtended by BC at P_{12} in two positions is the same.
3. From (2), $\angle 2 + \angle 3 + \angle 1 = \angle 1 + \angle 4 + \angle 5$

   or $\quad \angle 2 + \angle 3 = \angle 4 + \angle 5$

   i.e., B_1B_2 and C_1C_2 subtend equal angles at P_{12}.
4. P_{12} lies on the midnormal of B_1B_2,

   $\therefore \quad \angle 2 = \angle 3$

   Similarly, $\angle 4 = \angle 5$
5. $\quad \angle 2 + \angle 3 = \angle 4 + \angle 5$

   But $\angle 2 = \angle 3$ and $\angle 4 = \angle 5$,

$\therefore \quad \angle 2 = \angle 4$

and $\quad \angle 3 = \angle 5$

i.e., the input and the output links subtend equal angles at P_{12} in their corresponding positions.

6. $$\angle 2 + \angle 3 + \angle 1 = \angle 1 + \angle 4 + \angle 5$$
$$= \angle 1 + \angle 4 + \angle 3 \qquad (\angle 3 = \angle 5)$$

i.e., the angle subtended by the coupler link is equal to that subtended by the fixed pivots A and D.

7. The triangle $B_1P_{12}C_1$ moves as one link about P_{12} to the position $B_2P_{12}C_2$.
Angular displacement of coupler B_1C_1 = Angular displacement of $P_{12}C_1$ = Angular displacement of $P_{12}B_1$
i.e., $\quad \beta_{12} = \angle 4 + \angle 5 = \angle 2 + \angle 3$

5.2 RELATIVE POLE

A *pole* of a moving link is the centre of its rotation with respect to a fixed link. However, if the rotation of the link is considered relative to another moving link, the pole is known as the *relative pole*. The relative pole can be found by fixing the link of reference and observing the motion of the other link in the reverse direction.

For the four-link mechanism of Fig. 5.2, the pole of BC relative to AB is at the pivot B. The pole of DC relative to AB can be found as follows:

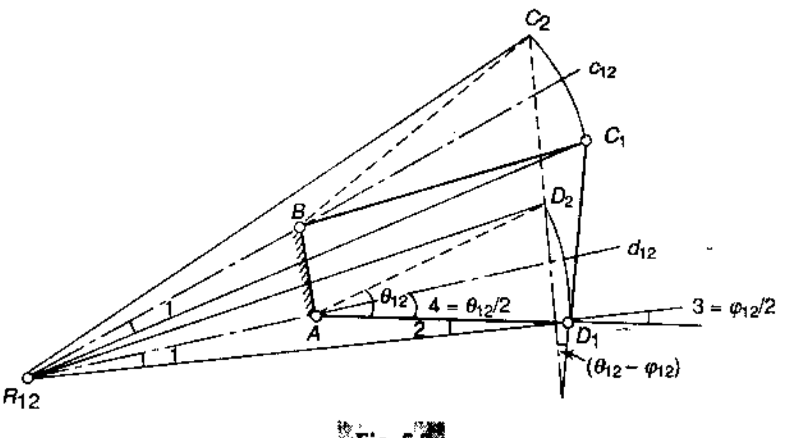


Fig. 5.2

Let θ_{12} = angle of rotation of AB (clockwise)

φ_{12} = angle of rotation of DC (clockwise)

Make the following constructions:

1. Assume A and B as the fixed pivots and rotate AD_1 about A through angle θ_{12} in the counter-clockwise direction (opposite to the direction of rotation of AB). Let D_2 be the new position after the rotation of AD (AB fixed).
2. Locate the point C_2 by drawing arcs with centres B and D_2 and radii equal to BC_1 and D_1C_1 respectively. Then ABC_2D_2 is known as the inversion of ABC_1D_1.
3. Draw midnormals of D_1D_2 and C_1C_2 which pass through A and B and intersect at R_{12} which is the required relative pole.

Now $(\varphi_{12} - \theta_{12})$ = Angle of rotation of the output link DC relative to the input link AB.

This angle is negative if $DC > AB$ and is positive if $DC < AB$.

Angular displacement of $R_{12}D_1$ = angular displacement of D_1C_1

[Refer Sec. 5.2 (7)]

$$\angle D_1R_{12}D_2 = -(\angle\varphi_{12} - \angle\theta_{12}) \qquad \text{(assuming } DC > AB\text{)}$$

or $$2\angle 1 = -(\angle\varphi_{12} - \angle\theta_{12})$$

or $$\angle 1 = -\frac{1}{2}(\angle\varphi_{12} - \angle\theta_{12})$$

In $\Delta AR_{12}D_1$, $$\angle 4 = \angle 1 + \angle 2$$

or $$\frac{1}{2}\angle\theta_{12} = -\frac{1}{2}(\angle\varphi_{12} - \angle\theta_{12}) + \angle 3 \qquad (\angle 2 = \angle 3)$$

$$= -\frac{1}{2}\angle\varphi_{12} + \frac{1}{2}\angle\theta_{12} + \angle 3$$

or $$\angle 3 = \frac{1}{2}\angle\varphi_{12}$$

The conclusion, just arrived, provides a method to locate the pole of the output link DC relative to the input link AB.

Procedure

1. Join A and D, the centres of the pivots.
2. Rotate AD about A through an angle $\theta_{12}/2$ in a direction opposite to that of AB.
3. Again rotate AD about D through an angle $\varphi_{12}/2$ in a direction opposite to that of DC. The point of intersection of the two positions of AD after rotation about A and D, is the required relative pole R_{12}. The angles subtended by D_1D_2 and C_1C_2 at R_{12} are the same,

i.e., $$\angle D_1R_{12}D_2 = \angle C_1R_{12}C_2$$
or $$2\,\angle D_1R_{12}A = 2\,\angle C_1R_{12}B$$
or $$\angle D_1R_{12}A = \angle C_1R_{12}B$$

Thus, it is also concluded that the angle subtended by the fixed pivots (A and D) at the relative pole is equal to the angle subtended by the coupler BC (Refer Sec. 5.1 also).

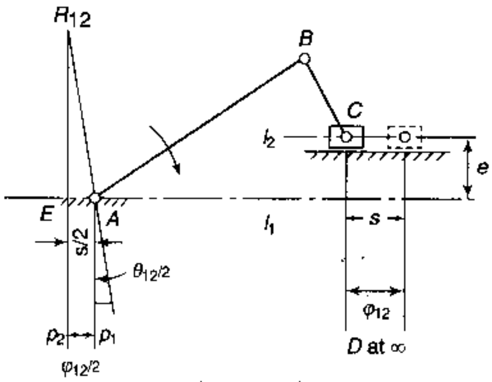


Fig. 5.3

Now, consider the slider-crank mechanism of Fig. 5.3. In this, if C reciprocates through a horizontal distance s, its centre of rotation will lie at infinity on a vertical line where the point D can also be assumed to lie. Then AD will also be a vertical line through A. Rotate AD about A through $\theta_{12}/2$ in the counter-clockwise direction as usual. Rotating AD about D through $\varphi_{12}/2$ would mean a vertical line towards the left of A, at a distance of $s/2$. The intersection of the two lines locates R_{12}.

Thus, the procedure to locate the relative pole of a slider-crank mechanism will be as under:

1. Draw two parallel lines l_1 and l_2 at a distance e apart (if there is an eccentricity).
2. Select a line segment AE of length $s/2$ on the line l_1 such that E is measured in a direction opposite to the motion of the slider.
3. At A and E, draw perpendicular lines p_1 and p_2 respectively to the line l_1
4. Make the angle $\theta_{12}/2$ at the point A with the line p_1 in a direction opposite to the rotation of the input link.

The intersection of this line with the line p_2 locates the relative pole R_{12}.

5.3 FUNCTION GENERATION BY RELATIVE POLE METHOD

The problems of function generation for two and three accuracy positions are easily solved by the relative pole method as discussed below:

(a) Four-link Mechanisms

Two-position synthesis Let for a four-link mechanism, the positions of the pivots A and D along with the angular displacements θ_{12} (angle between θ_1 and θ_2) and φ_{12} (angle between φ_1 and φ_2) of the driver and the driven links respectively be known.

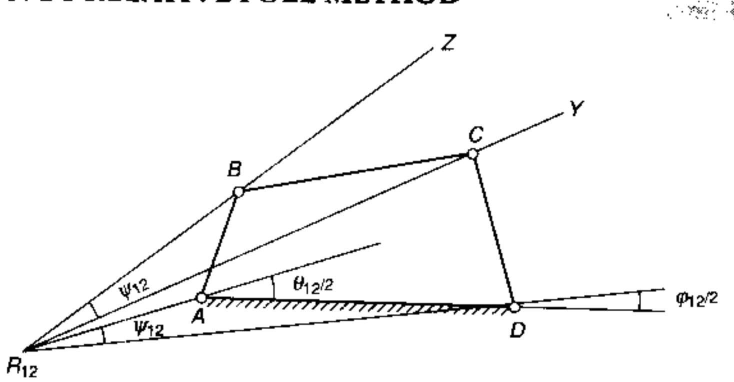


Fig. 5.4

To design the mechanism (Fig. 5.4), first locate the relative pole R_{12} by the procedure given in Sec. 5.2.

Now, angle subtended by the coupler BC at R_{12}

$$= \text{angle subtended by the fixed pivots } A \text{ and } D \text{ at } R_{12}$$

$$= \frac{1}{2}\angle\theta_{12} - \frac{1}{2}\angle\varphi_{12} \qquad \text{(assuming } DC > AB\text{)}$$

$$= \angle\psi_{12}$$

Adopt any of the following alternatives to design the required mechanism:

1. At point R_{12}, construct an angle ψ_{12} at an arbitrary position. Join any two points on the two arms of the angle to obtain the coupler link BC of the mechanism. Join AB and DC to have the driver and the driven links respectively.
2. Locate the point C arbitrary so that DC is the output link. Construct an angle $CR_{12}Z = \psi_{12}$. Take any point B on $R_{12}Z$. Join AB and BC.
3. Instead of locating the point C as above, locate the point B arbitrary so that AB is the input link. Construct an angle $BR_{12}Y = \psi_{12}$. Take any point C on $R_{12}Y$. Join BC and DC.

Then $ABCD$ is the required four-link mechanism.

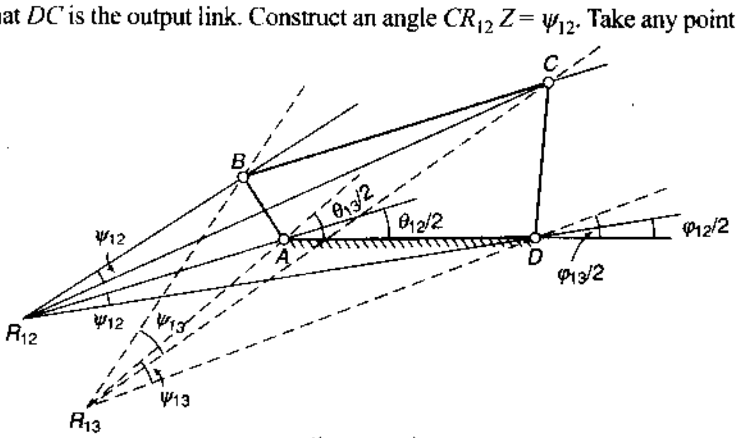


Fig. 5.5

Three-position synthesis If instead of one angular displacement of the input and of the output link, two displacements of the input (θ_{12} and θ_{13}) and two of the output (φ_{12} and φ_{13}) are known, find R_{12} and R_{13} as shown in Fig. 5.5.

Let ψ_{12} and ψ_{13} = angles made by the fixed link at R_{12} and R_{13} respectively.

Construct the angles ψ_{12} and ψ_{13} at the points R_{12} and R_{13} respectively in arbitrary positions such that the arms of the angles intersect at B and C in convenient positions.

(b) Slider-crank Mechanism

Two-position synthesis For a slider-crank mechanism,

let θ_{12} = angular displacement of input link ($\angle$between θ_1 and θ_2)

s_{12} = linear displacement of the slider

e = eccentricity

Draw two parallel lines l_1 and l_2 at a distance e apart. Locate the relative pole R_{12} as shown in Fig. 5.6. At the point R_{12}, construct an angle equal to $\theta_{12}/2$ ($\because \varphi_{12} = 0, \therefore \psi = \theta_{12}/2$) in an arbitrary (but convenient) position. The intersection of an arm of this angle with the line l_2 provides the position of the slider. Select an arbitrary point B on the other arm of the angle so that ABC is the required slider-crank mechanism.

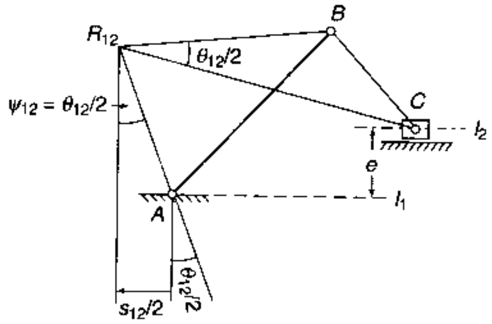


Fig. 5.6

Three-position synthesis If two displacements of the input link (θ_{12} and θ_{13}) and the slider (s_{12} and s_{13}) are known, find R_{12} and R_{13} as shown in Fig. 5.7.

Now, $\dfrac{\theta_{12}}{2}$ = angle made by the fixed link at R_{12}

$\dfrac{\theta_{13}}{2}$ = angle made by the fixed link at R_{13}

Therefore, construct angle $\theta_{12}/2$ at R_{12} in an arbitrary position locating the point C. Draw the angle $\theta_{13}/2$ at R_{13} with an arm along $R_{13}C$. Intersection of the two arms (not through C) of the two angles locates the point B.

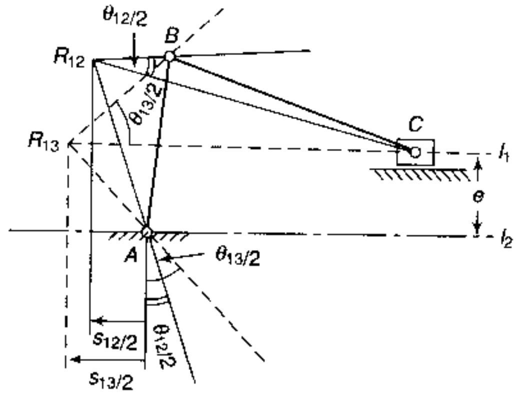
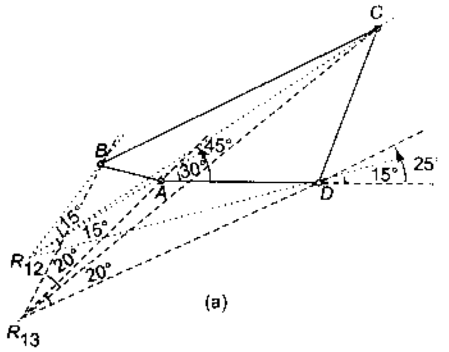


Fig. 5.7

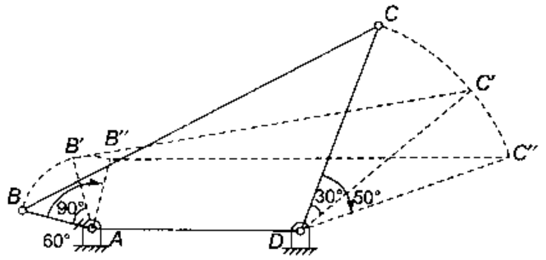
Example 5.1 *Design a four-link mechanism to coordinate three positions of the input and the output links for the following angular displacements:*

$\theta_{12} = 60°$ $\quad \varphi_{12} = 30°$

$\theta_{13} = 90°$ $\quad \varphi_{13} = 50°$



(a)



(b)

Fig. 5.8

Solution The procedure is as follows:

1. Locate suitable positions of the ground pivots A and D.
2. Locate the relative pole R_{12} by rotating AD about A through an angle 30° ($=\theta_{12}/2$)

[Fig. 5.8(a)] and about D through an angle 15° (= $\varphi_{12}/2$) taking both counter-clockwise. The point of intersection of the two positions of AD after rotation about A and D is the relative pole R_{12}. Similarly, locate R_{13}.

3. At point R_{12}, construct an angle of 15° (= $\theta_{12}/2 - \varphi_{12}/2$) at an arbitrary suitable position. At the point R_{13}, construct an angle of 20° (= $\theta_{13}/2 - \varphi_{13}/2$) in such a way that the intersection of its two arms with that of the arms of the previous angle locates points B and C at suitable positions.
4. Join AB, BC and CD.

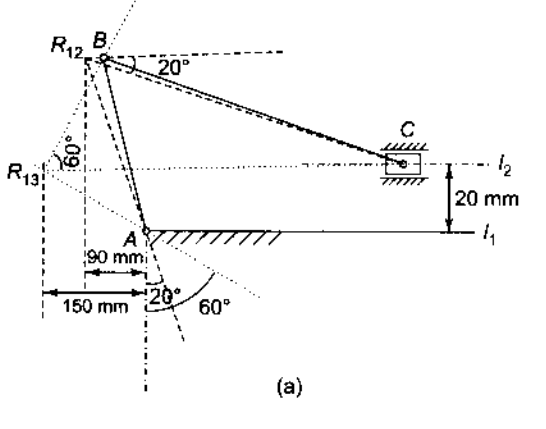
Then, $ABCD$ is the required four-link mechanism. Figure 5.8b shows the same in three positions.

Example 5.2 ***Design a slider-crank mechanism to coordinate three positions of the input link and the slider for the following angular and linear displacements of the input link and the slider respectively:***

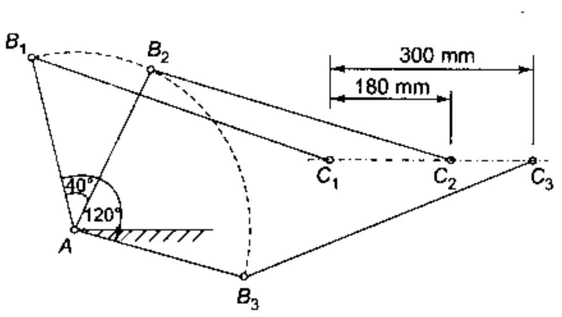
$\theta_{12} = 40°$ $\quad s_{12} = 180$ *mm*

$\theta_{13} = 120°$ $\quad s_{13} = 300$ *mm*

Take eccentricity of the slider as 20 mm.



(a)



(b)

Fig. 5.9

Solution The required slider-crank mechanism can be designed as follows:

1. Draw two parallel lines l_1 and l_2 20 mm apart from each other [Fig. 5.9(a)].
2. Take an arbitrary point A on the line l_1 for the fixed ground pivot.
3. Locate the relative pole R_{12} by rotating a vertical line through A about A through an angle of 20° (=$\theta_{12}/2$) counter-clockwise and drawing a vertical line at 90 mm (= $s_{12}/2$) to the left of A. Similarly, locate the relative pole R_{13} by rotating vertical line through A about A through an angle 60° (=$\theta_{13}/2$) counter-clockwise and drawing a vertical line at 150 mm (= $s_{13}/2$) to the left of A.
4. At point R_{12}, construct an angle of 20° (= $\theta_{12}/2$) and at point R_{13}, construct an angle of 60° (= $\theta_{13}/2$) in such a way that the intersection of their arms locate the points B and C (on l_2) at suitable positions.
5. Join AB and BC.

Then, ABC is the required slider-crank mechanism. Figure 5.9(b) shows the same in three positions.

5.4 INVERSION METHOD

Basically, the relative pole method is derived from the kinematic inversion principle. But there is no visible inversion of the planes during the solution of the problems. In the inversion method, there is direct use of the concept of inversion.

A four-link mechanism $ABCD$ is shown in two positions AB_1C_1D and AB_2C_2D in Fig. 5.10. The input and the output links AB and DC are moved through angles θ_{12} and φ_{12} respectively in the clockwise direction.

Rotate AD through θ_{12} in a direction opposite to the rotation of AB and get the inversion $AB_1C_2'\,D'$. It can be observed that the configuration AB_2C_2D has been rotated about A through an angle θ_{12} in the counter-clockwise direction to obtain the figure $AB_1C_2'D'$. Make the following observations:

1. Point C_2 is rotated through an angle θ_{12} in the counter-clockwise direction with the centre at A.
2. $C_1C'_2$ lies on a curve with the centre of rotation at B_1. Therefore, B_1 lies on the midnormal of $C_1C'_2$.

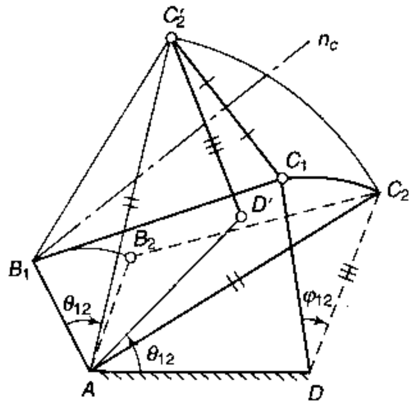


Fig. 5.10

5.5 FUNCTION GENERATION BY INVERSION METHOD

The problems of function generation for two, three and four accuracy positions can be solved by the inversion method as follows:

(a) Four-Link Mechanism

Two-position synthesis Let the distance between the fixed pivots A and D, and the angles θ_{12} and φ_{12} be known. To design the mechanism, proceed as follows:

1. Draw a line segment AD of length equal to the distance between the fixed pivots (Fig. 5.11).
2. At point D, construct an angle $C_1DC_2 = \varphi_{12}$ (clockwise) at an arbitrary position, selecting a suitable output crank length $DC_1 = DC_2$.
3. Rotate point C_2 in the counter-clockwise direction through an angle θ_{12} with A as centre and obtain the point C_2'.
4. Join C_1C_2' and draw its midnormal. Select a suitable point B_1 on it.
   AB_1C_1D is the required four-link mechanism in which B_1C_1 is the coupler.

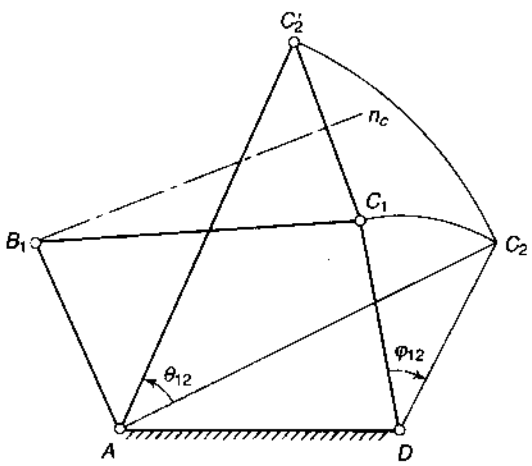


Fig. 5.11

Three-position synthesis If two angular displacements of the input link (θ_{12} and θ_{13}) and two of the output link (φ_{12} and φ_{13}) are known, proceed as below:

1. Draw a line segment AD of length equal to the distance between the fixed pivots (Fig. 5.12).
2. Choose some suitable length of the output link DC. Draw it at some suitable angle with the fixed link AD and locate its three positions DC_1, DC_2 and DC_3 as its angular displacements are known.
3. Find the points C'_2 and C'_3 after rotating AC_2 and AC_3 about A through angles θ_{12} and θ_{13} respectively in the counter-clockwise direction.
4. Intersection of the midnormals of $C_1 C'_2$ and $C_1 C'_3$ locates the point B_1. Then, AB_1C_1D is the required four-link mechanism.

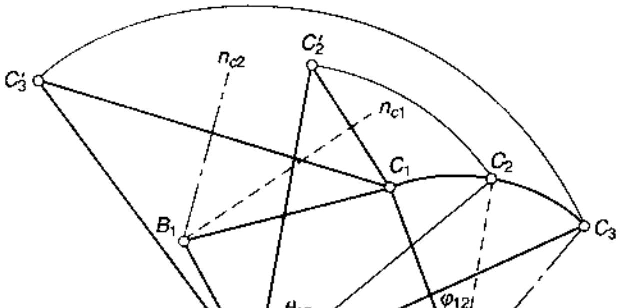


Fig. 5.12

The mechanism could also have been obtained by drawing the input link AB in three positions and rotating DB_2 and DB_3 through angles φ_{12} and φ_{13} respectively in the counter-clockwise direction with D as centre.

Four-position synthesis If a four-link mechanism is to be designed for four precision positions of the input and four positions of the output link, it can be designed by *point-position reduction* method. In this method, the point B_1 is chosen at the relative pole R_{12} with an assumed position of fixed link AD. The corresponding positions of B_2, B_3 and B_4 are easily located establishing the input link in four positions. Then by using the inversion method, the mechanism can be designed. The method is given below in brief:

1. Draw a line segment AD of suitable length to be the distance between the fixed pivots (Fig. 5.13).
2. Locate the position of the relative pole by rotating AD about A through angle $\theta_{12}/2$ and about D through an angle $\phi_{12}/2$ both in counter-clockwise direction. Take this as the point B_1.
3. Draw the input link AB in four positions AB_1, AB_2, AB_3 and AB_4 as its angular displacements are known.
4. Find the points B'_2, B'_3 and B'_4 after rotating DB_2, DB_3 and DB_4 about D through angles ϕ_{12}, ϕ_{13} and ϕ_{14} respectively in the counter-clockwise direction. It may be noted that the location of B'_2 is situated at B_1.
5. Intersection of the midnormals of $B'_2B'_3$ and $B'_3B'_4$ locates the point C.

Then AB_1CD is the required mechanism.

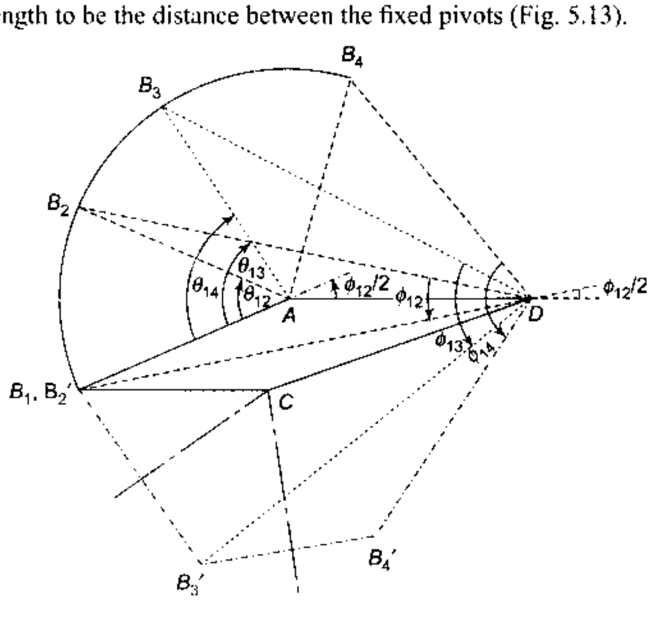


Fig. 5.13

(b) Slider-Crank Mechanism

Two-position synthesis If the angular displacement of the input link θ_{12} and the linear displacement of the slider s_{12} along with the eccentricity e are known, the required slider-crank mechanism is obtained as follows:

1. Draw two parallel lines l_1 and l_2 at a distance e apart (Fig. 5.14).
2. Take an arbitrary point A on the line l_1 for the fixed pivot and two points C_1 and C_2 on the line l_2, a distance s_{12} apart for the initial and the final positions of the slider.
3. Rotate the point C_2 about A through an angle θ_{12} in the counter-clockwise direction to obtain the point C'_2.
4. Join $C_1C'_2$ and draw its midnormal n_c. Take an arbitrary but convenient point B on it.

ABC_1 is the required slider-crank mechanism.

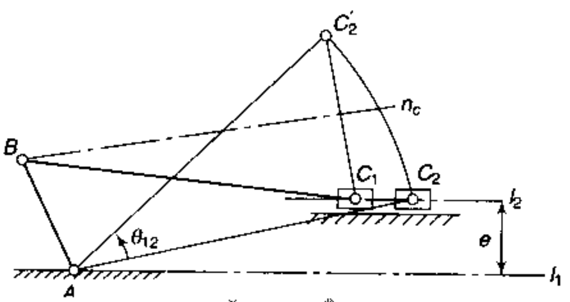


Fig. 5.14

Three-position synthesis For three positions of the input link and three positions of the slider, find C'_2 and C'_3 as usual. Then midnormal of $C_1C'_2$ and $C_1C'_3$ intersect at the point B (Fig.5.15).

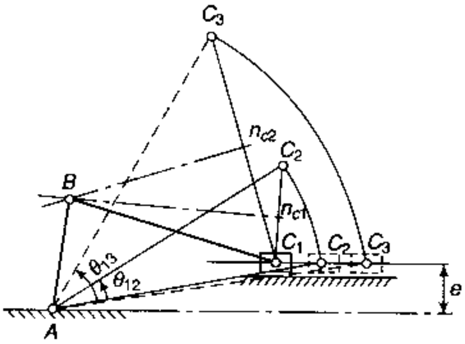


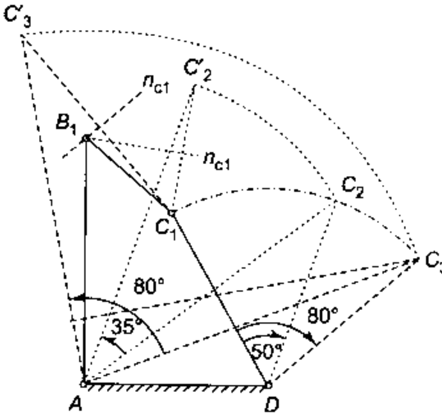
Fig. 5.15

Four-position synthesis A four-position synthesis can be done in the same way as in case of a four-link mechanism.

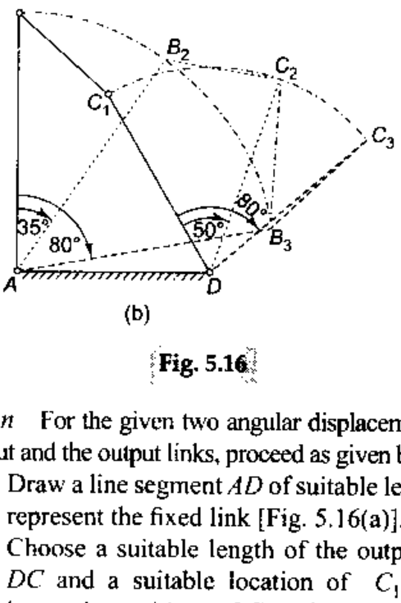
Example 5.3 *Design a four-link mechanism to coordinate three positions of the input and of the output links for the following angular displacements by inversion method:*

$\theta_{12} = 35°$ $\varphi_{12} = 50°$

$\theta_{13} = 80°$ $\varphi_{13} = 80°$



(a)



(b)

Fig. 5.16

Solution For the given two angular displacements of the input and the output links, proceed as given below:

1. Draw a line segment AD of suitable length to represent the fixed link [Fig. 5.16(a)].
2. Choose a suitable length of the output link DC and a suitable location of C_1. Then locate the positions of C_2 and C_3 by drawing the output link DC in three positions DC_1,

DC_2 and DC_3 as its angular displacements are known.

3. Find the points C'_2 and C'_3 after rotating AC_2 and AC_3 about A through angles θ_{12} and θ_{13} respectively in the counter-clockwise direction.
4. Intersection of the midnormals of $C_1\,C'_2$ and $C_1\,C'_3$ locates the point B_1.

Then AB_1C_1D is the required four-link mechanism. Figure 5.16(b) shows the mechanism in the required three positions. The mechanism could also have been obtained by drawing the input link AB in three positions as stated earlier.

Example 5.4 *Design a slider-crank mechanism to coordinate three positions of the input and of the slider for the following data by inversion method:*

$\theta_{12} = 30°$ $\qquad s_{12} = 40$ *mm*

$\theta_{13} = 60°$ $\qquad s_{13} = 96$ *mm*

Eccentricity = 20 mm

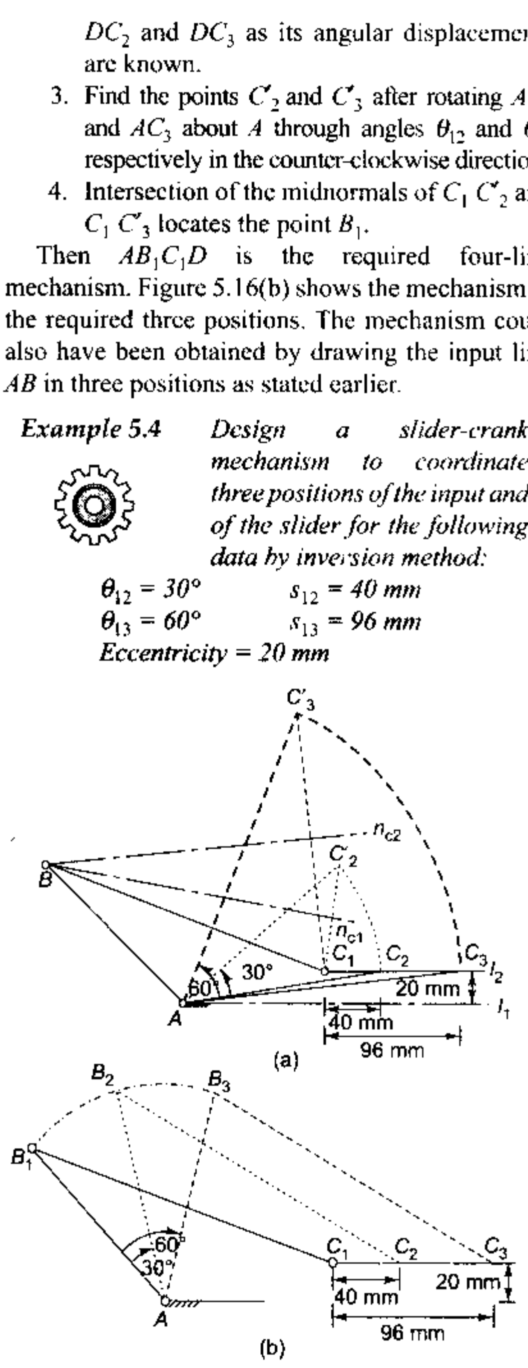


Fig. 5.17

Solution For the given two angular displacements of the input link and the two linear displacements of the slider along with the eccentricity e, the required slider-crank mechanism is obtained as follows:

1. Draw two parallel lines l_1 and l_2 at a distance of 20 mm apart [Fig. 5.17(a)].
2. Take an arbitrary point A on the line l_1 for the fixed pivot and three points C_1, C_2 and C_3 on the line l_2, at distances 40 mm and 96 mm apart for the initial and subsequent positions of the slider.
3. Rotate the point C_2 about A through an angle 30° in the counter-clockwise direction to obtain the point C'_2. Similarly, rotate the point C_3 about A through an angle 60° to obtain the point C'_3
4. Join $C_1C'_2$ and $C_1C'_3$ and draw their midnormals to intersect at point B.

Then ABC_1 is the required slider-crank mechanism. Figure 5.17(b) shows the mechanism in the required three positions.

Example 5.5 *Design a four-link mechanism to coordinate four positions of the input and the output links for the following angular displacements of the input link and the output link respectively:*

$\theta_{12} = 50°$ $\qquad \varphi_{12} = 30°$

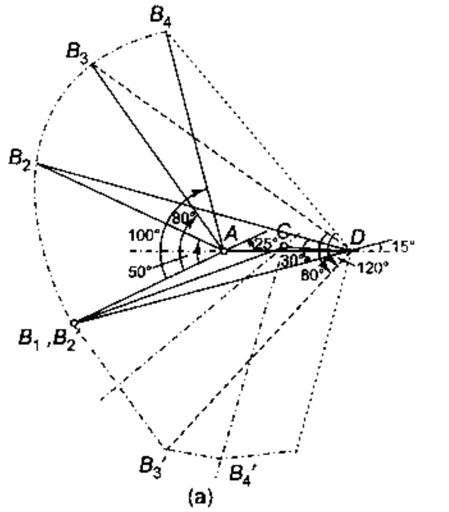
$\theta_{13} = 80°$ $\qquad \varphi_{13} = 80°$

$\theta_{13} = 100°$ $\qquad \varphi_{13} = 120°$

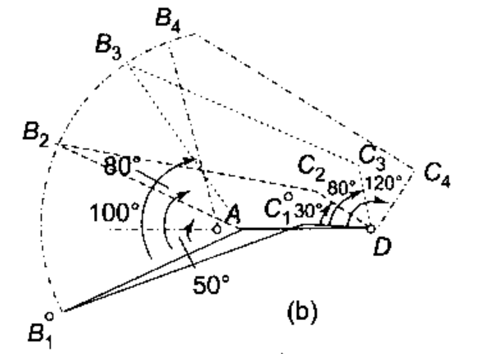
Solution Make the following construction:

1. Draw a line segment AD of suitable length to be the distance between the fixed pivots [Fig. 5.18(a)].
2. Locate the position of the relative pole by rotating AD about A through angle 25° (= $\theta_{12}/2$) and about D through an angle 15° (= $\varphi_{12}/2$) both in counter-clockwise direction. Take this as point B_1.

3. Draw the input link AB in four positions AB_1, AB_2, AB_3 and AB_4 as its angular displacements are known.



(a)



(b)

Fig. 5.18

4. Locate the points B'_2, B'_3 and B'_4 by rotating DB_2, DB_3 and DB_4 about D through angles φ_{12}, φ_{13} and φ_{12} respectively in the counter-clockwise direction such that the location of B'_2 is at B_1.
5. Intersection of the midnormals of $B'_2B'_3$ and $B'_3B'_4$ locates the point C.

Then AB_1CD is the required mechanism. Figure 5.18(b) shows the mechanism in the required four positions.

5.6 PATH GENERATION

The problem may be of designing the mechanism without or with prescribed timing, i.e., the guidance of the point on the coupler may or may not be coordinated with the movement of the input link. To design such a mechanism, the method of inversion of mechanisms is used by fixing the coupler and releasing the fixed link. To understand the inversion method, consider a four-link mechanism as shown in Fig. 5.19(a) in two positions $A_1B_1C_1D_1$ and $A_1B_2C_2D_1$. E is an offset point on the coupler which assumes the location E_2 in the second position. In Fig. 5.19(b), the inversion of the mechanism is shown by fixing the coupler B_1C_1 and releasing the fixed link so that the quadrilateral $A_1B_2C_2D$ of figure (a) in exactly the same as the quadrilateral $A_2B_1C_1D_2$ of figure (b). It can be done by taking $\angle A_2\ B_1C_1 = \angle A_1B_2C_2$. Now if triangles $B_2E_2D_1$ and $B_1E_1D_2$ are marked in the two figures, they must be congruent. It can be observed that the point C_1 is the centre of curvature of the arc passing through D_1 and D_2 and thus lies on the right bisector of D_1D_2.

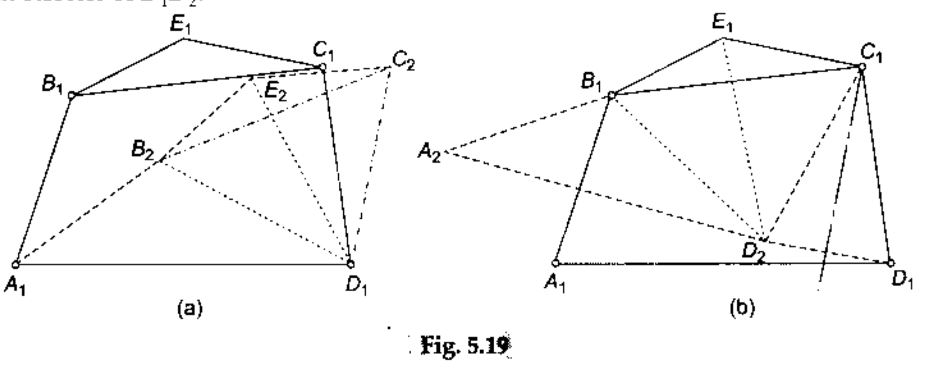


Fig. 5.19

(a) Without prescribed timing In this case, three positions of the coupler point (E_1, E_2, E_3) are known. The procedure of designing such a mechanism is as follows:

1. Select suitable locations of A_1 and D_1 of the fixed link with respect to the positions of the coupler point E_1, E_2 and E_3 (Fig. 5.20).
2. Choose a suitable length of the input link A_1B_1. Mark the first position of B_1 at a suitable position. As the lengths of the links A_1B_1 and B_1E_1 are to be same in all positions, locate the positions of B_2 and B_3. Thus, the input link in three positions AB_1, AB_2 and AB_3 is established. Now, the task of obtaining the point C_1 remains which is done by the inversion method as discussed above by fixing the coupler in the first position.
3. Construct $\Delta E_2B_2D_1 \equiv \Delta E_1B_1D_2$ and $\Delta E_3B_3D_1 \equiv \Delta E_1B_1D_3$.
4. The centre of the arc through D_1, D_2 and D_3 is the crank pin C_1. Draw midnormals of D_1D_2 and D_2D_3. The intersection of the two locations is the point C_1.

   Thus, $A_1B_1C_1D_1$ is the required four-link mechanism with the coupler point E_1.

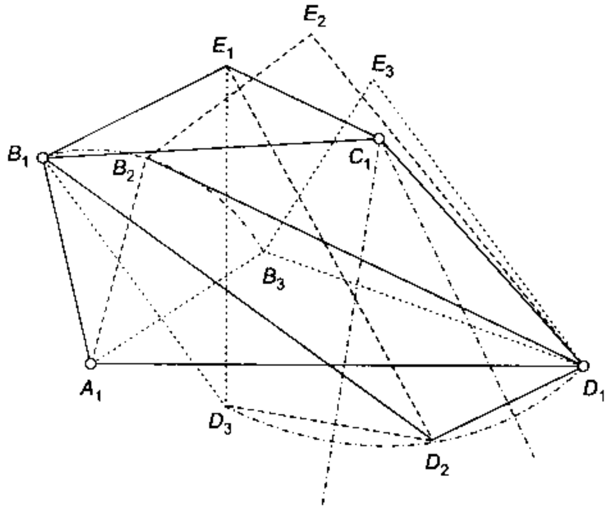


Fig. 5.20

(b) With prescribed timing For two angular displacements θ_{12} and θ_{13} and three positions of the coupler point (E_1, E_2, E_3), the choice of the input link is not arbitrary. To design such a mechanism, proceed as follows (Fig. 5.21).

1. Select suitable locations of A_1 and D_1 of the fixed link with respect to the positions of the coupler point (E_1, E_2, E_3).
2. Rotate AE_2 through an angle θ_{12} in the counter-clockwise direction with A as centre and obtain the point E'_2. Similarly, rotate AE_3 through an angle θ_{13} in the counter-clockwise direction with centre A and obtain the point E'_3.
3. The centre of the arc through E_1, E'_2 and E'_3 is the crank pin B_1. To obtain it, draw midnormals of $E_1E'_2$ and $E_1E'_3$, the intersection of these provides the location of B_1.
4. The rest of the procedure is as discussed above for the case of without prescribed timing, i.e., locating the positions of B_2 and B_3 and then constructing $\Delta E_2B_2D_1 \equiv \Delta E_1B_1D_2$ and $\Delta E_3B_3D_1 \equiv \Delta E_1B_1D_3$.

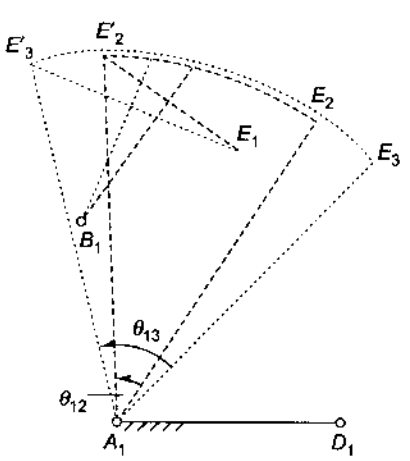
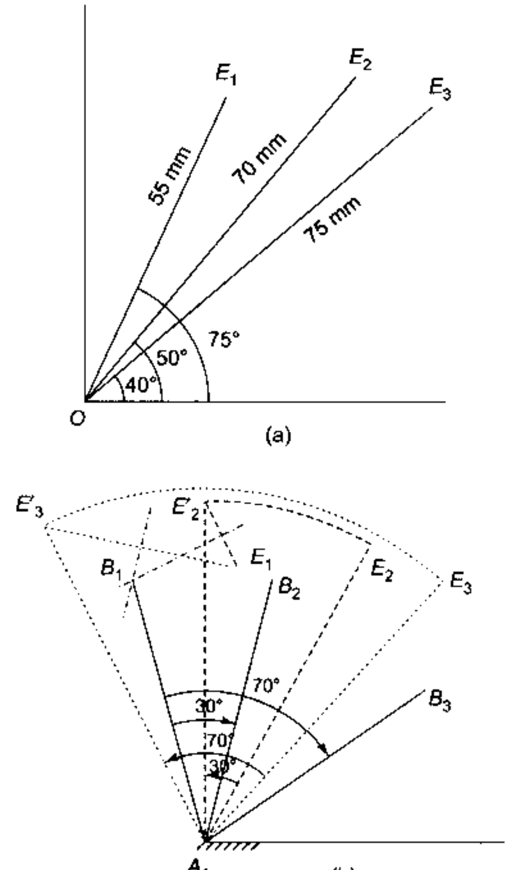


Fig. 5.21

Example 5.6 *Design a four-link mechanism to coordinate the following three positions of the coupler point. The positions are given with respect to coordinate axes:*

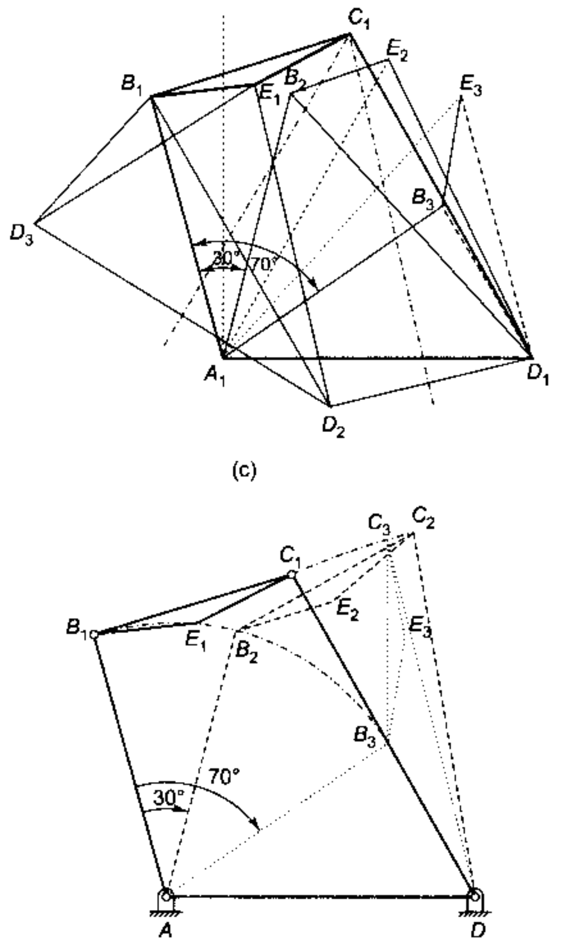
$r_1 = 55\ mm$	$\alpha_1 = 75°$
$r_2 = 70\ mm$	$\alpha_2 = 50°$
$r_3 = 75\ mm$	$\alpha_3 = 40°$

The angular displacements of the input link are to be $\theta_{12} = 30°$ and $\theta_{13} = 70°$.



(a)

(b)



(c)

(d)

Fig. 5.22

Solution It is the case of path generation with prescribed timing. The procedure is given below:

1. Locate the three coupler points E_1, E_2 and E_3 as shown in Fig. 5.22(a).
2. Select a suitable location of the pivot A_1 of the fixed link with respect to the positions of the coupler points [Fig. 5.22(b)].
3. Rotate A_1E_2 through an angle 30° ($=\theta_{12}$) in the counter-clockwise direction with A_1 as centre and obtain the point E'_2. Similarly, rotate AE_3 through an angle 70° ($=\theta_{13}$) in the counter-clockwise direction with centre A and obtain the point E'_3.

4. Draw midnormals of $E_1E'_2$ and $E_1E'_3$, the intersection locates the point B_1.
5. Draw the input link in three positions AB_1, AB_2 and AB_3.
6. Select suitable location of the pivot D_1 of the fixed link. Construct $\Delta E_2B_2D_1 \equiv \Delta E_1B_1D_2$ and $\Delta E_3B_3D_1 \equiv \Delta E_1B_1D_3$.
7. Draw midnormals of D_1D_2 and D_2D_3. The intersection of the two locates the point C_1.

Thus, $A_1B_1C_1D_1$ is the required four-link mechanism with the coupler point E_1. Figure 5.22(d) shows the required mechanism in three positions.

5.7 MOTION GENERATION (RIGID-BODY GUIDANCE)

Let a rigid body be guided through three prescribed positions. It is required to design a four-link mechanism of which this rigid body will be a coupler. The rigid body is shown in Fig. 5.23 in three given positions. To find the lengths of the four links of the mechanism, proceed as follows:

1. Take any two arbitrary suitable points B and C on the rigid body and locate these on the body in three positions. It is assumed that the point B_1, B_2, B_3 and E_1, E_2 and E_3 are non-collinear.
2. Find the centre A of the circle passing through B_1, B_2, and B_3. Similarly, let the centre of the circle passing through C_1, C_2 and C_3 be D.
3. Join AB_1, B_1C_1 and C_1D.

Then, AB_1C_1D is the required mechanism which takes the coupler B_1C_1 through B_2C_2 and B_3C_3.

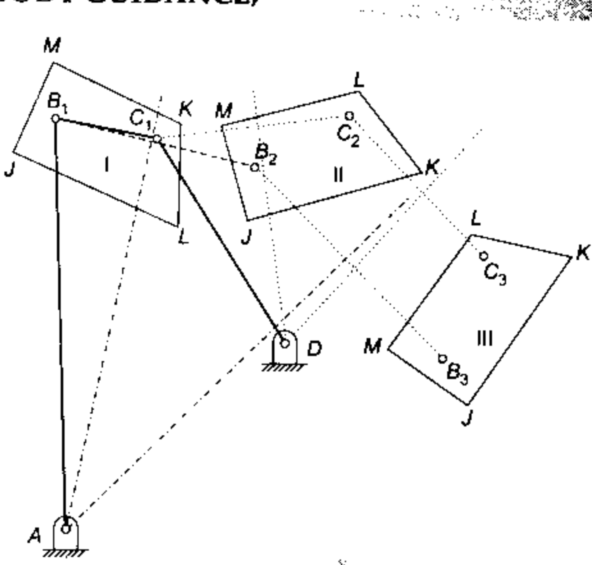


Fig. 5.23

In the above case of motion generation, the choice of the ground pivots is not with the designer. Many times, it becomes necessary to fix the locations of these pivots beforehand due to constraint of space. Such type of problem can also be solved by the inversion method as discussed in Section 5.6. In such cases, proceed as follows:

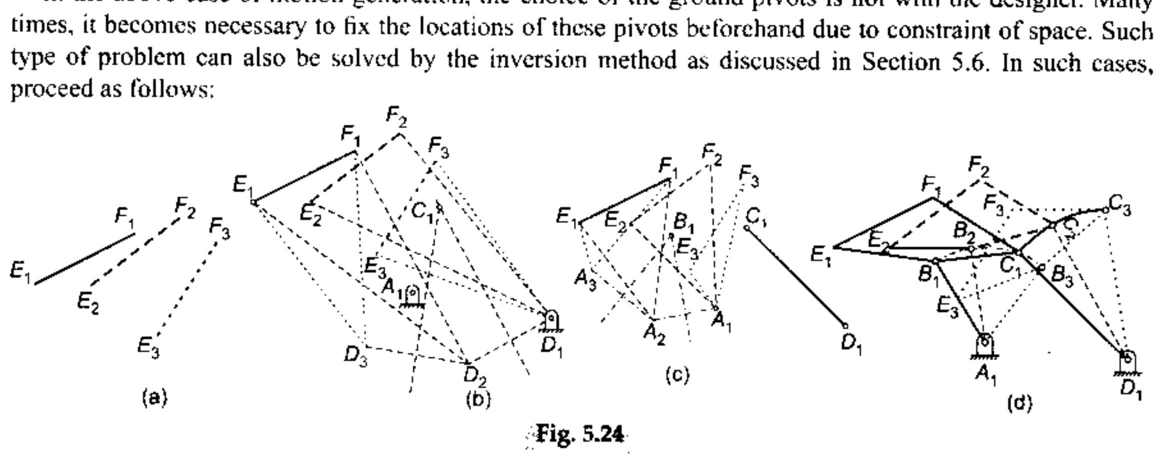


Fig. 5.24

1. Take any two arbitrary points E and F on the rigid body and locate these on the body in three positions [Fig. 5.24(a)].
2. Let A_1 and D_1 be the locations of the ground pivots [Fig. 5.24(b)].
3. Construct $\Delta E_2F_2D_1 \equiv \Delta E_1F_1D_2$ and $\Delta E_3F_3D_1 \equiv \Delta E_1F_1D_3$.
4. The centre of the arc through D_1, D_2 and D_3 is the crank pin C_1. To locate it, draw midnormals of D_1D_2 and D_2D_3. The intersection of the two is the pivot point C_1 on the rigid body or the coupler.
5. Construct $\Delta E_2F_2A_1 \equiv \Delta E_1F_1A_2$ and $\Delta E_3F_3A_1 \equiv \Delta E_1F_1A_3$ [Fig. 5.24(c)].
6. The centre of the arc through A_1, A_2 and A_3 is the crank pin B_1. Draw midnormals of A_1A_2 and A_2A_3. The intersection of the two locates the pivot point A_1 on the rigid body or coupler.

Then $A_1B_1C_1D_1$ is the required mechanism which takes the coupler $B_1E_1F_1C_1$ through $B_2E_2F_2C_2$ and $B_3E_3F_3C_3$ [Fig. 5.24(d)].

PART B: COMPUTER-AIDED SYNTHESIS OF MECHANISMS

5.8 FUNCTION GENERATION

A four-link mechanism shown in Fig. 5.25 is in equilibrium. Let a, b, c and d be the magnitudes of the links AB, BC, CD and DA respectively. θ, β and φ are the angles of AB, BC and DC respectively with the X-axis (taken along AD). AD is the fixed link. AB and DC are the input and output links respectively of the mechanism.

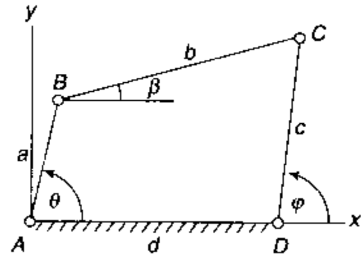


Fig. 5.25

Considering the links to be vectors, displacement along the X-axis

$a\cos\theta + b\cos\beta = d + c\cos\phi$ (The equation is valid for < ϕ more than 90° also.)

or $\quad b\cos\beta = c\cos\varphi - a\cos\theta + d$

or $\quad (b\cos\beta)^2 = (c\cos\varphi - a\cos\theta + d)^2$

$$= c^2\cos^2\varphi + a^2\cos^2\theta + d^2 - 2ac\cos\theta\cos\varphi - 2ad\cos\theta + 2cd\cos\varphi \qquad \text{(i)}$$

Displacement along Y-axis

$a\sin\theta + b\sin\beta = c\sin\phi$

or $\quad b\sin\beta = c\sin\varphi - a\sin\theta$

or $\quad (b\sin\beta)^2 = (c\sin\varphi - a\sin\theta)^2$

$$= c^2\sin^2\varphi + a^2\sin^2\theta - 2ac\sin\theta\sin\varphi \qquad \text{(ii)}$$

Adding (i) and (ii),

$$b^2 = c^2 + a^2 + d^2 - 2ac\cos\theta\cos\varphi - 2ad\cos\theta + 2cd\cos\varphi - 2ac\sin\theta\sin\varphi$$

or $\quad 2cd\cos\varphi - 2ad\cos\theta + a^2 - b^2 + c^2 + d^2 = 2ac(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$

Dividing throughout by $2ac$,

$$\frac{d}{a}\cos\varphi - \frac{d}{c}\cos\theta + \frac{a^2 - b^2 + c^2 + d^2}{2ac} = \cos(\theta - \varphi) = \cos(\varphi - \theta)$$

This is known as *Freudenstein's* equation and can be written as,

$$k_1\cos\varphi + k_2\cos\theta + k_3 = \cos(\theta - \varphi) \qquad (5.1)$$

where

$$k_1 = \frac{d}{a};\ k_2 = -\frac{d}{c};\ \text{and}\ \ k_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

Let the input and the output are related by some function such as $y = f(x)$ and for the specified positions

$\theta_1, \theta_2, \theta_3$ = three positions of input link (given)

and $\varphi_1, \varphi_2, \varphi_3$ = three positions of output link (given)

It is required to find the values of a, b, c and d to form a four-link mechanism giving the prescribed motions of the input and the output links.

Equation (5.1) can be written as,

$$k_1\cos\varphi_1 + k_2\cos\theta_1 + k_3 = \cos(\theta_1 - \varphi_1)$$
$$k_1\cos\varphi_2 + k_2\cos\theta_2 + k_3 = \cos(\theta_2 - \varphi_2)$$
$$k_1\cos\varphi_3 + k_2\cos\theta_3 + k_3 = \cos(\theta_3 - \varphi_3)$$

k_1, k_2, and k_3 can be evaluated by Gaussian elimination method or by the Cramer's rule.

$$\Delta = \begin{vmatrix} \cos\varphi_1 & \cos\theta_1 & 1 \\ \cos\varphi_2 & \cos\theta_2 & 1 \\ \cos\varphi_3 & \cos\theta_3 & 1 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} \cos(\theta_1 - \varphi_1) & \cos\theta_1 & 1 \\ \cos(\theta_2 - \varphi_2) & \cos\theta_2 & 1 \\ \cos(\theta_3 - \varphi_3) & \cos\theta_3 & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} \cos\varphi_1 & \cos(\theta_1 - \varphi_1) & 1 \\ \cos\varphi_2 & \cos(\theta_2 - \varphi_2) & 1 \\ \cos\varphi_3 & \cos(\theta_3 - \varphi_3) & 1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} \cos\varphi_1 & \cos\theta_1 & \cos(\theta_1 - \varphi_1) \\ \cos\varphi_2 & \cos\theta_2 & \cos(\theta_2 - \varphi_2) \\ \cos\varphi_3 & \cos\theta_3 & \cos(\theta_3 - \varphi_3) \end{vmatrix}$$

k_1, k_2 and k_3 are given by,

$$k_1 = \frac{\Delta_1}{\Delta}; \quad k_2 = \frac{\Delta_2}{\Delta}; \quad k_3 = \frac{\Delta_3}{\Delta}$$

Knowing k_1, k_2 and k_3, the values of a, b, c and d can be computed from the relations

$$k_1 = \frac{d}{a}; \quad k_2 = -\frac{d}{c}; \quad k_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$$

Value of either a or d can be assumed to be unity to get the proportionate values of other parameters.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
    float a,b,c,p1,p2,p3,t1,t2,t3,th2,th3,a1,a2,a3,del,
    rad,ph1,ph2,ph3,del1,del2,del3;
    clrscr();
    printf("enter values of th1,th2,th3,ph1,ph2,ph3;\n");
    scanf("%f%f%f%f%f%f",&th1,&th2,&th3,&ph1,&ph2,&ph3);
```

```
        rad=4*atan(1)/180;
        p1=cos(ph1*rad);
        p2=cos(ph2*rad);
        p3=cos(ph3*rad);
        t1=cos(th1*rad);
        t2=cos(th2*rad);
        t3=cos(th3*rad);
        a1=cos((th1-ph1)*rad);
        a2=cos((th2-ph2)*rad);
        a3=cos((th3-ph3)*rad);
        del=p1*(t2-t3)+t1*(p3-p2)+(p2*t3-p3*t2);
        del1=a1*(t2-t3)+t1*(a3-a2)+(a2*t3-a3*t2);
        del2=p1*(a2-a3)+a1*(p3-p2)+(p2*a3-p3*a2);
        del3=p1*(t2*a3-t3*a2)+t1*(a2*p3-a3*p2)+a1*(p2*t3p3*t2);
        a=del/del1;
        c=-del/del2;
              b=pow((a*a+c*c+1-2*a*c*del3/del),.5);
              printf("  a       b  c       d\n");
        printf("%6.2f %6.2f %6.2f %6.2f \n",a,b,c,1.00);
        getch();
}
```

Fig. 5.26

Figure 5.26 shows a program in C for solving such a problem. The input variables are

`th1, th2, th3`	Angular displacements of the input link (degrees)
`ph1, ph2, ph3`	Angular displacements of the output link (degrees)

The output variables are

`a, b, c, d`	Ratio of magnitudes of the links *AB*, *BC*, *CD* and *AD* respectively.

Least-square Technique

The above synthesis technique is used to synthesize a mechanism where three finitely separated positions of the input and the output links are known. It is observed that a four-link mechanism can be designed precisely up to five positions of the input and the output links, provided θ and φ are measured from some arbitrary reference. In such cases, the synthesis equations become non-linear and have to be solved by using other means than the Cramer's rule.

It is not possible to design a mechanism for more than five positions of the input and the output links. However, it is possible to design a mechanism which gives least deviation from the specified positions and provides the average performance. To achieve this, an approximated solution of the problem is sought which gives the least error. A method known as the *least-square technique* is useful in synthesizing such a mechanism.

Considering Freudenstein's equation,

$$k_1\cos\varphi_i + k_2\cos\varphi_i + k_3 - \cos(\theta_i - \varphi_i) = 0$$

Owing to error, this equation is not satisfied. Its LHS will have some error value. As this can be positive or negative, its square is taken and summed up for n values of θ and φ and defining.

$$S = \sum_{i=1}^{n} [k_1 \cos\varphi_i + k_2 \cos\theta_i + k_3 - \cos(\theta_i - \varphi_i)]^2$$

Conditions for this to be minimum are,

$$\frac{\partial S}{\partial k_1} = 0, \frac{\partial S}{\partial k_2} = 0 \text{ and } \frac{\partial S}{\partial k_3} = 0$$

i.e.
$$\sum_{i=1}^{n} 2[k_1 \cos\varphi_i + k_2 \cos\theta_i + k_3 - \cos(\theta_i - \varphi_i)]\cos\varphi_i = 0$$

or

$$k_1 \sum k_1 \cos^2 \varphi_i + k_2 \sum \cos\theta_i \cos\varphi_i + k_3 \sum \cos\varphi_i = \sum \cos(\theta_i - \varphi_i)\cos\varphi_i \qquad (5.2)$$

Similarly,

$$k_1 \sum \cos\varphi_i \cos\theta_i + k_2 \sum \cos^2 \theta_i + k_3 \sum \cos\theta_i = \sum \cos(\theta_i - \varphi_i)\cos\theta_i \qquad (5.3)$$

and
$$k_1 \sum \cos\varphi_i + k_2 \sum \cos\theta_i + k_3 \sum 1 = \sum \cos(\theta_i - \varphi_i) \qquad (5.4)$$

These are three simultaneous linear, homogenous equations in three unknowns k_1, k_2, and k_3. These can be solved by using Cramer's rule or other means.

Figure 5.27 shows a program to find the ratio of different links using the least-square technique. The input variables are

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
     int i,k;
     float a,b1,b2,b3,tt,b,c,d,p1,p2,p3,t1,t2,t3,th1,th2,
     th3,a1,a2,a3,del,rad,ph1,ph2,ph3,del1,del2,del3;
     float th[10],ph[10];
     clrscr();

     printf("enter i the number of positions\n");
     scanf("%d",&i);
     printf("enter i values of th[i] and ph[i]\n");
     for(k=0;k<i;k++)     scanf("%f",&th[k]);
     for(k=0;k<i;k++)     scanf("%f",&ph[k]);
     rad=4*atan(1)/180;
       for(k=0;k<i;k++)
       {
            p1=p1-pow(cos(ph[k]*rad),2);
            p2=p2+(cos(th[k]*rad))*(cos(ph[k)*rad));
            p3=p3+cos(ph[k]*rad);
            t1=p2;
            t2=t2+(cos(th[k]*rad)*(cos(th[k]*rad);
            t3=t3+cos(th[k)*rad);
            b1=p3;
            b2=t3;
            b3=i;
            tt=cos((th[k)-ph[k))*rad);
            a1=a1+tt*cos(ph[k]*rad);
```

```
            a2=a2+tt*cos(th[k]*rad);
            a3=a3+tt;
      }
del=p1*(t2*b3-t3*b2)+t1*(b2*p3-b3*p2)+b1*(p2*t3-p3*t2);
del1=a1*(t2*b3-t3*b2)+t1*(b2*a3-b3*a2)+b1*(a2*t3-a3*t2);
del2=p1*(a2*b3-a3*b2)+a1*(b2*p3-b3*p2)+b1*(p2*a3-p3*a2);
del3=p1*(t2*a3-t3*a2)+t1*(a2*p3-a3*p2)+a1*(p2*t3-p3*t2);
a=del/del1;
c=-del/del2;
b=sqrt(a*a+c*c+1-2*a*c*del3/del);
printf(" a      b       c     d\n");
printf("%6.2f %6.2f %6.2f %6.2f \n",a,b,c,1.00);
getch();
```

Fig. 5.27

`i`	Number of specified positions
`th(j)`	Angular displacements of the input link AB for j = 1 to i (in degrees)
`ph(j)`	Angular displacements of the output link DC for j = 1 to i (in degrees)

Example 5.7 ***Design a four-link mechanism to coordinate three positions of the input and the output links as follows:***

$\theta_1 = 20°, \varphi_1 = 35°$
$\theta_2 = 35°, \varphi_2 = 45°$
$\theta_3 = 50°, \varphi_3 = 60°$

Solution

$k_1 \cos 35° + k_2 \cos 20° + k_3 = \cos(20° - 35°)$

$k_1 \cos 45° + k_2 \cos 35° + k_3 = \cos(35° - 45°)$

$k_1 \cos 60° + k_2 \cos 50° + k_3 = \cos(50° - 60°)$

Now,

$$\Delta = \begin{vmatrix} \cos 35° & \cos 20° & 1 \\ \cos 45° & \cos 35° & 1 \\ \cos 60° & \cos 50° & 1 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} \cos(20° - 35°) & \cos 20° & 1 \\ \cos(35° - 45°) & \cos 35° & 1 \\ \cos(50° - 60°) & \cos 50° & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} \cos 35° & \cos(20° - 35°) & 1 \\ \cos 45° & \cos(35° - 45°) & 1 \\ \cos 60° & \cos(50° - 60°) & 1 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} \cos 35° & \cos 20° & \cos(20° - 35°) \\ \cos 45° & \cos 35° & \cos(35° - 45°) \\ \cos 60° & \cos 50° & \cos(50° - 60°) \end{vmatrix}$$

$\Delta = \cos 35°(\cos 35° - \cos 50°) + \cos 20°(\cos 60° - \cos 45°) + (\cos 45° \times \cos 50° - \cos 60° \times \cos 35°)$
$= -0.005\ 204$

Similarly, $\Delta_1 = -0.003\ 33$
$\Delta_2 = 0.003\ 910\ 6$
$\Delta_3 = -0.005\ 973\ 5$

Assuming $d = 1$,

$$k_1 = \frac{\Delta_1}{\Delta} = \frac{-0.003\,33}{-0.005\,204} = \frac{1}{a} \text{ or } a = 1.56$$

$$k_2 = \frac{\Delta_1}{\Delta} = \frac{0.003\,910\,6}{-0.005\,204} = -\frac{1}{c} \text{ or } c = 1.33$$

$$k_3 = \frac{\Delta_1}{\Delta} = \frac{-0.005\,973\,5}{-0.005\,204} = \frac{1.56^2 - b^2 + 1.33^2 + 1^2}{2 \times 1.56 \times 1.33}$$

or $b = 0.66$

Thus, a, b, c and d are 1.56, 0.66, 1.33 and 1.00 respectively.

The input and the output using the program of Fig. 5.26 have been shown in Fig. 5.28.

```
Enter values of th1, th2, th3, ph1, ph2, ph3;
20  35  50  35  45  60
```

a	b	c	d
1.56	0.66	1.33	1.00

Fig. 5.28

The mechanism is shown in Fig. 5.29.

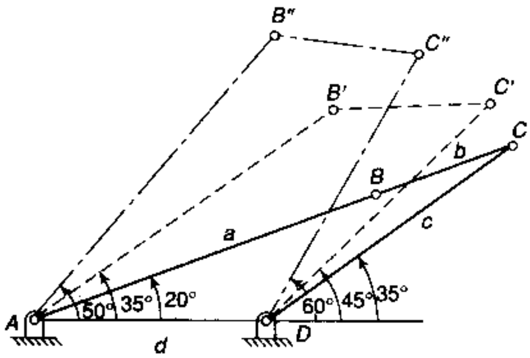


Fig. 5.29

Example 5.8 ***Design a four-link mechanism when the motions of the input and the output links are governed by a function $y = x^2$ and x varies from 0 to 2 with an interval of 1. Assume θ to vary from 50° to 150° and ϕ from 80° to 160°.***

Solution The angular displacement of the input link is governed by x whereas that of the output link, by y.

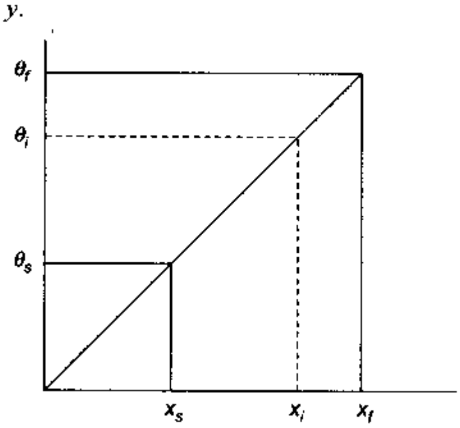


Fig. 5.30

Let subscripts s, f and i indicate the start, final and any value in the range.

Range of $x = x_f - x_s = 2 - 0 = 2$ and thus

$x_1 = 0; \quad x_2 = 1; \quad x_3 = 2$

The corresponding values of y are according to function, $y = x^2$

Range of $y = y_f - y_s = 4 - 0 = 4$ and ;

$y_1 = 0; \quad y_2 = 1; \quad y_3 = 4$

Range of $\theta = \theta_f - \theta_s = 150° - 50° = 100°$

Range of $\phi = \phi_f - \phi_s = 160° - 80° = 80°$

Refer Fig. 5.30 which indicates a linear relationship between x and θ. Thus

$$\frac{\theta_i - \theta_s}{\theta_f - \theta_s} = \frac{x_i - x_s}{x_f - x_s}$$

or

$$\theta_i = \theta_s + \frac{\theta_f - \theta_s}{x_f - x_s}(x_i - x_s) = \theta_s + \frac{\Delta\theta}{\Delta x}(x_i - x_s);$$

Thus, $\theta_1 = 50° + \frac{100°}{2} \times 0 = 50°$;

$\theta_2 = 50° + \frac{100°}{2} \times 1 = 100°$;

$\theta_3 = 50° + \frac{100°}{2} \times 2 = 150°$

Similarly,

$$\varphi_i = \varphi_s + \frac{\varphi_f - \varphi_s}{y_f - y_s}(y_i - y_s) = \varphi_s + \frac{\Delta\varphi}{\Delta y}(y_i - y_s);$$

or $\varphi_1 = 80° + \frac{80°}{4} \times 0 = 80°$;

$\varphi_2 = 80° + \frac{80°}{4} \times 1 = 100°$;

$\varphi_3 = 80° + \frac{80°}{4} \times 4 = 160°$

This can be written in a tabular form:

Position	x	y	θ	φ
1	**0**	0	**50°**	80°
2	**1**	1	**100°**	100°
3	2	4	**150°**	160°

Thus, we have the following equations,

$k_1 \cos 80° + k_2 \cos 50° + k_3 = \cos 30°$

$k_1 \cos 100° + k_2 \cos 100° + k_3 = \cos 0°$

$k_1 \cos 160° + k_2 \cos 150° + k_3 = \cos 10°$

Using Cramer's rule,

$\Delta = -0.3850$

$\Delta_1 = -0.1052 \quad k_1 = 0.273 = \frac{d}{a}$

$\Delta_2 = 0.1079 \quad k_2 = -0.280 = -\frac{d}{c}$

$\Delta_3 = 0.3844 \quad k_3 = 0.988 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$

which gives

$a = 3.66$ units
$b = 1.02$ units
$c = 3.57$ units
and $d = 1$ unit

Figure 5.31 shows the required mechanism.

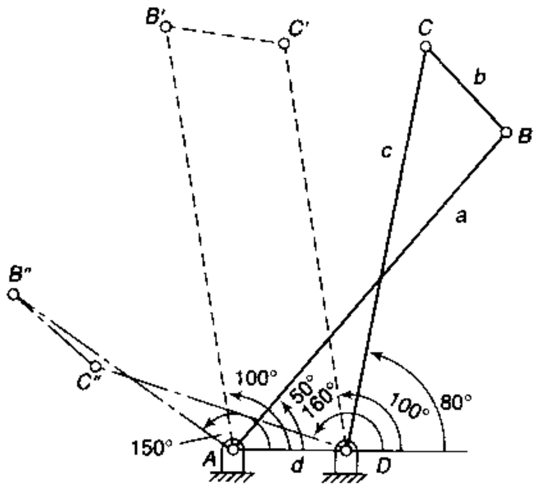


Fig. 5.31

Example 5.9 *Design a four-link mechanism when the motions of the input and the output links are governed by a function $y = 2\log_{10} x$ and x varies from 2 to 4 with an interval of 1. Assume θ to vary from 30° to 70° and ϕ from 40° to 100°.*

Solution Let subscripts s, f and i indicate the start, final and any value in the range.

Range of $x = x_f - x_s = 4 - 2 = 2$ and thus

$x_1 = 2; \quad x_2 = 3; \quad x_3 = 4$

The corresponding values of y are according to function, $y = 2\log_{10} x$

Range of $y = y_f - y_s = (2\log_{10}4) - (2\log_{10}2) = 1.204 - 0.602 = 0.602$

and $y_1 = 0.602; \quad y_2 = 2\log_{10} 3 = 0.954; \quad y_3 = 1.204;$

Range of $\theta = \theta_f - \theta_s = 70° - 30° = 40°$

Range of $\phi = \phi_f - \phi_s = 100° - 40° = 60°$

As $\theta_1 = \theta_s = 30°$ and $\theta_3 = \theta_f = 70°$, there is no need of finding them.

$\theta_i = \theta_s + \frac{\Delta\theta}{\Delta x}(x_i - x_s)$ and thus

$\theta_2 = 30° + \frac{40°}{2} \times 1 = 50°$

Similarly, As $\varphi_1 = \varphi_s = 40°$ and $\varphi_3 = \varphi_f = 100°$, there is no need of finding them.

$\varphi_2 = 40° + \frac{60°}{0.602}(0.954 - 0.602) = 75°$

This can be written in a tabular form also.

Position	x	y	θ	φ
1	2	0.602	30°	40°
2	3	0.954	50°	75°
3	4	1.204	70°	100°

Thus, we have the following equations,

$k_1 \cos 40° + k_2 \cos 30° + k_3 = \cos(-10°)$
$k_2 \cos 75° + k_2 \cos 50° + k_3 = \cos(-25°)$
$k_3 \cos 100° + k_2 \cos 70° + k_3 = \cos(-30°)$

Using Cramer's rule,

$\Delta = 0.0560$

$\Delta_1 = 0.0146 \qquad k_1 = 0.2607 = \frac{d}{a}$

$\Delta_2 = -0.0135 \qquad k_2 = -0.241 = -\frac{d}{c}$

$\Delta_3 = 0.0557$

$k_3 = 0.995 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}$

which gives

$a = 3.83$ units
$b = 1.14$ units
$c = 4.14$ units
and $d = 1$ unit

Figure 5.32 shows the required mechanism.

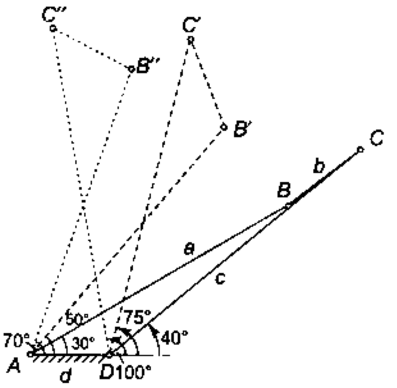


Fig. 5.32

Example* 5.10** ***Design a four-link mechanism to coordinate the motion of the input and the output links governed by a function y = log x for 0 < x ≤ 8. Take δx = 1. The range for θ is from 15° to 120° whereas for, φ it is from 20° to 150°.

Solution The angular positions of the input and the output links are tabulated below:

X	y	θ	φ
1	0	15°	20°
2	0.69	30°	*63°
3	1.10	45°	89°
4	1.39	60°	107°
5	1.61	75°	121°
6	1.79	90°	132°
7	1.95	105°	142°
8	2.08	120°	150°

$$* 20° + (150° - 20°) \times \frac{0.69}{2.08} = 63°$$

It is required to design the mechanism so that the input and the output links pass through eight specified positions. It is not possible to design such a mechanism. However, using the least-square technique, a mechanism may be devised which gives the least deviation from the specified positions.

The dimensions of various links are shown in Fig. 5.33 using the program given in Fig. 5.27.

```
Enter i the number of specified positions 8
Enter i values of th[i] and ph[i]
  15  30  45  60  75  90  105  120
  20  63  89  107  121  132  142  150
  a        b          c    d
  2.42   0.91      2.37 1.00
```

Fig. 5.33

Figure 5.34 shows the required mechanism which will give the least deviation from the specified positions.

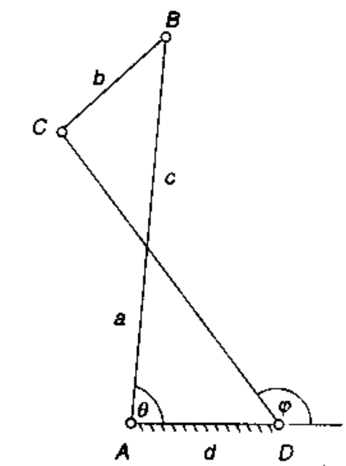


Fig. 5.34

CHEBYCHEV SPACING

In function-generation problems, the output is related to the input through a function $y = f(x)$ and it is required to obtain the dimensions of a linkage to satisfy this relationship. In general, a linkage synthesis problem does not have exact solution over its entire range of travel. However, it is usually possible to design a linkage which exactly satisfies the desired function at a few chosen positions known as *precision* or *accuracy* *points* or *positions*. It is assumed that the design deviates very slightly from the desired function between the precision positions and that the deviation is within acceptable limit. The difference between the function prescribed and the function produced by the designed linkage is known as the *structural error*. For most of the cases, this error may be about 3 to 4 per cent.

The amount of structural error also depends upon the choice of the precision points. A judicious use of precision points greatly affects the structural error. Thus, a set of precision points may be selected for use in the synthesis of the linkage which can minimize the structural error and a fair choice is provided by *Chebychev spacing*. For n accuracy positions in the range $x_o \le x \le x_{n+1}$, the Chebychev spacing is given by

$$x_i = \frac{x_{n+1} + x_o}{2} - \frac{x_{n+1} - x_o}{2} \cos \frac{(2i-1)\pi}{2n} \quad \text{where } i = 1, 2, 3 \ldots n$$

For example, if it is desired to design a linkage to satisfy the function $y = \sqrt{x}$ over the range $1 \le x \le 3$ using three precision positions, then the three values of x are

$$x_1 = \frac{3+1}{2} - \frac{3-1}{2} \cos \frac{(2-1)\pi}{2 \times 3} = 2 - \cos \frac{\pi}{6} = 1.134$$

$$x_2 = 2 - \cos \frac{3\pi}{6} = 2$$

$$x_3 = 2 - \cos \frac{5\pi}{6} = 2.866$$

And the corresponding values of y, $\quad y_1 = 1.065; \quad y_1 = 1.414; \quad y_1 = 1.693$

Graphical approach Chebychev spacing of accuracy points can also be found easily by the graphical method. The method is as follows:

1. Draw a circle of diameter equal to the range Δx $(= x_{n+1} - x_o)$.
2. Inscribe a regular polygon of $2n$ sides in the circle such that the two sides of the polygon are perpendicular to the x-axis.
3. Draw projections of the vertices of the polygon on the x-axis. The perpendiculars intersect the diameter Δx at the precision points.

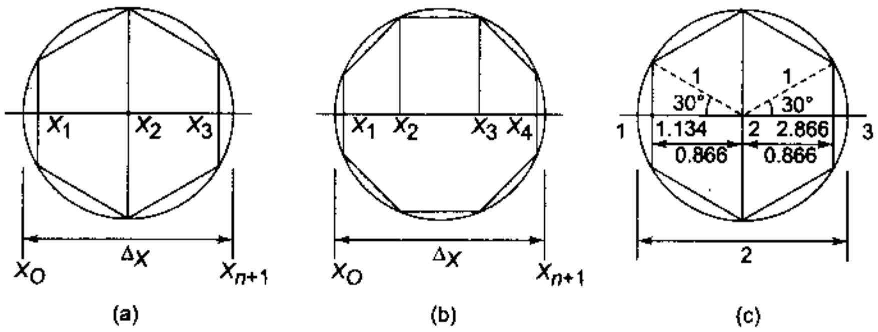


Fig. 5.35

Figure 5.35(a) and (b) shows the graphical method for $n = 3$ and $n = 4$ respectively. Figure 5.35(c) shows the construction for the above example.

Example 5.11 ***Design a four-link mechanism if the motions of the input and the output links are governed by a function $y = x^{1.5}$ and x varies from 1 to 4. Assume θ to vary from 30° to 120° and φ from 60° to 130°. The length of the fixed link is 30 mm. Use Chebychev spacing of accuracy points.***

Solution

$$x_i = \frac{x_{n+1} + x_o}{2} - \frac{x_{n+1} - x_o}{2}\cos\frac{(2i-1)\pi}{2n}$$

$$x_1 = \frac{4+1}{2} - \frac{4-1}{2}\cos\frac{(2-1)\pi}{2\times 3} = 2.5 - 1.5\cos\frac{\pi}{6} = 1.2$$

$$x_2 = 2.5 - 1.5\cos\frac{3\pi}{6} = 2.5$$

$$x_3 = 2.5 - 1.5\cos\frac{5\pi}{6} = 3.8$$

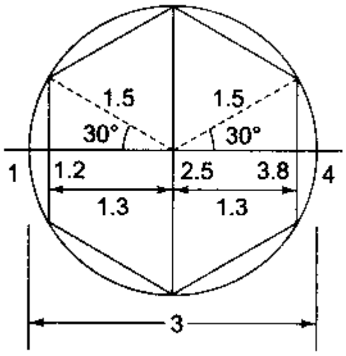


Fig. 5.36

Figure 5.36 shows Chebychev spacing of accuracy points by graphical method.

Let subscripts s and f indicate the start and final values in the range.

The corresponding values of y,

$y_1 = 1.315$; $y_1 = 3.953$; $y_1 = 7.408$;

Also, $y_s = 1^{1.5} = 1$; and $y_f = 4^{1.5} = 8$

Range of $x = x_f - x_s = 4 - 1 = 3$

Range of $y = y_f - y_s = 8 - 1 = 7$

$$\theta_1 = 30^\circ + \frac{120^\circ - 30^\circ}{4-1}(1.2-1) = 36^\circ;$$

$$\varphi_1 = 60^\circ + \frac{130^\circ - 60^\circ}{8-1}(1.315-1) = 63.2^\circ;$$

$$\theta_2 = 30^\circ + \frac{90^\circ}{3}(2.5-1) = 75^\circ;$$

$$\varphi_2 = 60^\circ + \frac{70^\circ}{7}(3.953-1) = 89.5^\circ;$$

$$\theta_3 = 30^\circ + \frac{90^\circ}{3}(3.8-1) = 114^\circ;$$

$$\varphi_3 = 60^\circ + \frac{70^\circ}{7}(7.408-1) = 124.1^\circ;$$

Now, $k_1\cos 63.2^\circ + k_2\cos 36^\circ + k_3 = \cos(36^\circ - 63.2^\circ) = \cos 27.2^\circ$

$k_1\cos 89.5^\circ + k_2\cos 75^\circ + k_3 = \cos(75^\circ - 89.5^\circ) = \cos 14.5^\circ$

$k_1\cos 124.1^\circ + k_2\cos 114^\circ + k_3 = \cos(114^\circ - 124.1^\circ) = \cos 10.1^\circ$

Solving by Cramer's rule,

$k_1 = 2.286$; $k_2 = -1.98$; $k_3 = 1.461$

Now, $d = 30$ mm

$k_1 = \frac{30}{a} = 2.286$ or $a = 13.1$ mm

$k_2 = -\frac{30}{c} = -1.98$ or $c = 15.2$ mm

$k_3 = \frac{13.1^2 - b^2 + 15.2^2 + 30^2}{2\times 13.1\times 15.2}$ or $b = 26.8$ mm

Thus, a, b, c and d are 13.1 mm, 26.5 mm, 15.2 mm and 30 mm respectively.

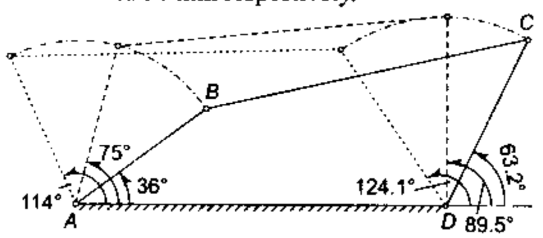


Fig. 5.37

The mechanism is shown in Fig. 5.37 in three positions.

5.30 PATH GENERATION

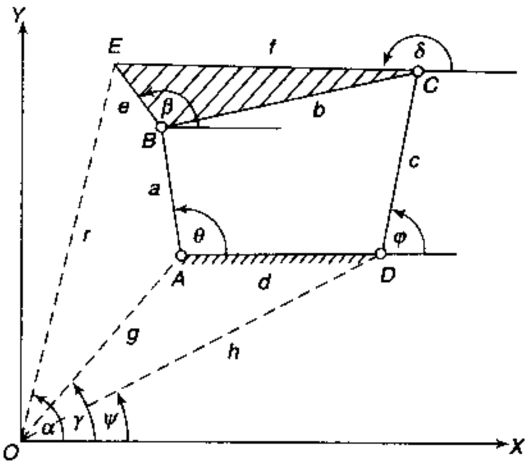


Fig. 5.38

A four-link mechanism $ABCD$ with a coupler point E is shown in Fig. 5.38. Three positions of the input link (θ_1, θ_2, θ_3) and three positions of the coupler point E given by three values of r and α, i.e., r_1, r_2, r_3 and α_1, α_2, α_3 are known. It is required to find the dimensions of a, c, e and f along with the location of the pivots A and D given by g, γ and h, ψ respectively so that the coupler point E generates the specified path with the motion of the input link AB.

For the loop $OABE$, considering the links to be vectors

$$g \cos \gamma + a \cos \theta + e \cos \beta - r \cos \alpha = 0 \quad (5.5)$$

and $$g \sin \gamma + a \sin \theta + e \sin \beta - r \sin \alpha = 0 \quad (5.6)$$

or $$e \cos \beta = r \cos \alpha + g \cos \gamma - a \cos \theta$$

and $$e \sin \beta = r \sin \alpha + g \sin \gamma - a \sin \theta$$

Squaring and adding,

$$e^2 = r^2 + g^2 + a^2 - 2gr (\cos \alpha \cos \gamma + \sin \alpha \sin \gamma)$$

$$+ 2ag (\cos \theta \cos \gamma + \sin \theta \sin \gamma) - 2ar (\cos \theta \cos \alpha + \sin \theta \sin \alpha)$$

or $$2ar \cos (\theta - \alpha) + 2gr \cos (\alpha - \gamma) + (e^2 - a^2 - g^2) = r^2 + 2ag \cos (\theta - \gamma)$$

or $$2ar \cos (\theta - \alpha) + 2gr \cos (\alpha - \gamma) + k = r^2 + 2ag \cos (\theta - \gamma) \quad (5.7)$$

where

$$k = e^2 - a^2 - g^2 \quad (5.8)$$

Inserting the values of r_1, r_2, r_3; α_1, α_2, α_3 and θ_1, θ_2, θ_3, we obtain three equations. The unknowns are a, g, e and γ. Thus, for three equations, there are four unknowns and therefore, equations cannot be solved. However, the value of one of the unknown can be assumed. Assuming the value of γ, we are left with three unknowns a, g, e and there are three equations to solve them.

Even now, the equations cannot be solved as such, as these are non-linear equations. However, by making the following substitutions, these can be solved easily.

Let

$$\left.\begin{aligned} a &= l_a + \lambda m_a \\ g &= l_g + \lambda m_g \\ k &= l_k + \lambda m_k \end{aligned}\right\} \quad (5.9)$$

where
$$\begin{aligned} \lambda &= ag \\ &= (l_a + \lambda m_a)(l_g + \lambda m_g) \\ &= l_a l_g + \lambda l_a m_g + \lambda l_g m_a + \lambda^2 m_a m_g \end{aligned}$$

or $$m_a m_g \lambda^2 + (l_a m_g + l_g m_a - 1)\lambda + l_a l_g = 0$$

or $$A\lambda^2 + B\lambda + C = 0$$

or

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (5.10)$$

where

$$A = m_a m_g$$

$$B = l_a m_g + l_g m_a - 1$$
$$C = l_a l_g$$

Thus, Eq. (5.7) becomes,

$$2(l_a + \lambda m_a) r \cos(\theta - \alpha) + 2(l_g + \lambda m_g) r \cos(\alpha - \gamma) + l_k + \lambda m_k = r^2 + 2\lambda \cos(\theta - \gamma)$$

Separating the components into two groups; one with and the other without λ,

$$l_a[2r\cos(\theta - \alpha)] + l_g[2r\cos(\alpha - \gamma)] + l_k = r^2 \quad (5.11)$$

$$m_a[2r\cos(\theta - \alpha)] + m_g[2r\cos(\alpha - \gamma)] + m_k = 2\cos(\theta - \gamma) \quad (5.12)$$

From Eq. (5.11), three equations can be written as,

$$l_a[2r_1\cos(\theta_1 - \alpha_1)] + l_g[2r_1\cos(\alpha_1 - \gamma)] + l_k = r_1^2 \quad (5.13)$$

$$l_a[2r_2\cos(\theta_2 - \alpha_2)] + l_g[2r_2\cos(\alpha_2 - \gamma)] + l_k = r_2^2 \quad (5.14)$$

$$l_a[2r_3\cos(\theta_3 - \alpha_3)] + l_g[2r_3\cos(\alpha_3 - \gamma)] + l_k = r_3^2 \quad (5.15)$$

These are three linear equations l_a, l_g and l_k and can be solved by applying Cramer's rule or by other means.

Similarly, m_a, m_g, and m_k can also be found.

As l_a, l_g, l_k and m_a, m_g, m_k have been found, a, g and k can be calculated from the relations of Eq. (5.9).

Also, $e = \sqrt{k + a^2 + g^2}$ [from Eq. (5.8)]

From Eq. (5.5), three values of β can be found,

$$e\cos\beta = r\cos\alpha - g\cos\gamma - a\cos\theta$$

$$\beta_1 = \cos^{-1}\left[\frac{r_1\cos\alpha_1 - g\cos\gamma - a\cos\theta_1}{e}\right] \quad (5.16)$$

Similarly, β_2 and β_3 can be found.

Thus, we have obtained the values of a, e, g, γ and β. The whole procedure can be repeated for the loop *ODCE*. The following equations are formed,

$$h\cos\psi + c\cos\varphi + f\cos\delta - r\cos\alpha = 0 \quad (5.17)$$

$$h\sin\psi + c\sin\varphi + f\sin\delta - r\sin\alpha = 0 \quad (5.18)$$

These equations are similar to Eqs (5.5) and (5.6).

Assuming
$$f = l_f + \lambda' m_f$$
$$h = l_h + \lambda' m_h$$
$$p = l_p + \lambda' m_p$$

Two sets of equations similar to Eqs (5.11) and (5.12) are obtained by eliminating ϕ as given below:

$$l_f[2r\cos(\delta - \alpha)] + l_k[2r\cos(\alpha - \psi)] + l_p = r^2 \quad (5.19)$$

$$m_f[2r\cos(\delta - \alpha)] + m_h[2r\cos(\alpha - \psi)] + m_p = 2\cos(\delta - \psi) \quad (5.20)$$

In these equations, α and r are known. ψ can be assumed. Also, assuming δ_1, the values of δ_2 and δ_3 can be found as follows:

The angular displacements of the coupler link *BCE* is the same at the points *B* and *C*,

$$\delta_2 - \delta_1 = \beta_2 - \beta_1$$

or
$$\delta_2 = \delta_1 + (\beta_2 - \beta_1) \quad (5.21)$$

Similarly,

$$\delta_3 = \delta_1 + (\beta_3 - \beta_1) \quad (5.22)$$

Solving the Eqs (5.19) and (5.20), the values of f, h and c can be known.

As the points *A*, *B*, *E*, *C* and *D* are located, the dimensions *a*, *b*, *c*, *d*, *e* and *f* can be obtained. Figure 5.39 shows a program for the solution of such a problem.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
(
    FILE*fp;
    int k,j;
    float al,a2,a3,all,a22,a33,a12,a21,g12,g21,e12,e21,akl,
    ak2,ak3,all,al2,al3,aa,gl,g2,g3,tll,t22,t33,tbl,tb2,tb3,
    gg,gamm,ss,si,dll,d22,d33,rl,r2,r3,pl,p2,p3,tl,t2,t3,
    cl,c2,c3,alg,ala,alk,ama,amg,amk,bb,cc,all,el,e2,squ,
    betl,bet2,bet3,pll,p22,p33,e3,gs,del,dell,del2,del3,rad;
    clrscr();
    printf("Enter values of tbl,tb2,tb3,rl,r2,r3,all,all,
    al2,al3,");
    printf("gamm,si,dell\n");
    scanf("%f%f%f%f%f%f%f%f%f%f%f%f",&tbl,&tb2,&tb3,&rl,
    &r2,&r3,&all,&al2,&al3,&gamm,&si,&dell);
    rad=4*atan(1)/180;
    tll=tbl*rad;
    t22=tb2*rad;
    t33=tb3*rad;
    all=all*rad;
    a22=al2*rad;
    a33=al3*rad;
    gg=gamm*rad;
    ss=si*rad;
    dll=dell*rad;
     for(j=0;j<3;j-H)
     {
        pl=2*rl*cos(tll-all);
        p2=2*r2*cos(t22-a22);
        p3=2*r3*cos(t33-a33);
        tl=2*rl*cos(all-gg);
        t2=2*r2*cos(a22-gg);
        t3=2*r3*cos(a33-gg);
        cl=rl*rl;
        c2=r2*r2;
        c3=r3*r3;
            for(k=0;k<2;k++)
            (
            del=pl*(t2-t3)+tl*(p3-p2)+(p2*t3-p3*t2);
            dell=cl*(t2-t3)+tl*(c3-c2)+(c2*t3-c3*t2);
            del2=pl*(c2-c3)+cl*(p3-p2)+(p2*c3-p3*c2);
            del3=pl*(t2*c3-t3*c2)+tl*(c2*p3-c3*p2);
            +cl*(p2*t3-p3*t2);
            akl=dell/del;
```

```
 ak2=del2/del;
 ak3=del3/del;
 if(k==0)
 {
        ala=ak1;
        alg=ak2;
        alk=ak3;
        c1=2*cos(t11-gg);
        c2=2*cos(t22-gg);
        c3=2*cos(t33-gg);
        }
 }
ama=ak1;
amg=ak2;
amk=ak3;
aa=ama*amg;
bb=ala*amg+alg*ama-1;
cc=ala*alg;
squ=bb*bb-4*aa*cc;
if(squ>0)
 {
 all=sqrt(squ);
 al1=(-bb-all)/(2*aa);
 al2=(-bb+all)/(2*aa);
 a1=ala+al1*ama;
 g1=alg+al1*amg;
 a2=ala+al2*ama;
 g2=alg+al2*amg;
 e1=sqrt(alk+al1*amk+a1*a1+g1*g1);
 e2=sqrt(alk+al2*amk+a2*a2+g2*g2);
 if(j==0){printf("   g      a      e");
 printf("     h      c      f\n"); }
 if(j==1){printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f
 \n",g12,a12,e12,g1,e1,a1);
 printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f\n",
 g12,a12,e12,g2,e2,a2); }
 if (j==2) {printf("%8.2f %8.2f %8.2f %8.2f %8.2f
 %8.2f \n",g21,a21,e21,g1,e1,a1);
 printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f \n",
 g21,a12,e21,g2,e2,a2); }
 if(j==0)
 {
        g12=g1;
        a12=a1;
        e12=e1;
        g21=g2;
        a21=a2;
        e21=e2;
        gs=gg;
        p11=t11;
```

```
                p22=t22;
                p33=t33;
            }
            if(j==1)
            {
                g1=g21;
                a1=a21;
                e1=e21;
                t11=d11;
                t22=d22;
                t33=d33;
                gg=gs;
                t11=p11;
                t22=p22;
                t33=p33;
                }
            bet1=acos((r1*cos(a11)-g1*cos(gg)-a1*cos(t11))/e1);
            bet2=acos((r2*cos(a22)-g1*cos(gg)-a1*cos(t22»/e1);
            bet3=acos((r3*cos(a33)-g1*cos(gg)-a1*cos(t33»)/e1);
            d22=d11+bet2-bet1;
            d33=d11+bet3-bet1;
            a3=a2;
            g3=g2;
            e3=e2;
            p11=t11;
            p22=t22;
            p33=t33;
            t11=d11;
            t22=d22;
            t33=d33;
            gs=gg;
            gg=ss;
        }
    }
getch();
}
```

Fig. 5.3

The input variables are:

`tb1, tb2, tb3`	angular displacement of the input link AB (degrees)
`r1, r2, r3`	radial distances of the coupler point from origin (mm)
`al1, al2, al3`	angular position of the coupler point (degrees)
`gamm`	assumed value of the angle γ (degrees)
`si`	assumed value of the angle ψ (degrees)
`del1`	assumed value of the angle δ_1 (degrees)

The output variables are

`g, a, e, h, c, f`	distances or lengths of the links in mm

If more than three positions of the input link along with the same number of positions of the coupler point are known, the mechanism can be synthesized using the least-square technique. The deviations of the coupler point E from the prescribed positions will be minimum in such a design. Thus,

$$S_1 = [\,l_a\{2r\cos(\theta - \alpha)\} + l_g\{2r\cos(\alpha - \gamma)\} + l_k - r^2]^2$$

$$S_2 = [\,m_a\{2r\cos(\theta - \alpha)\} + m_g\{2r\cos(\alpha - \gamma)\} + m_k - 2\cos(\theta - \alpha)]^2$$

For minimum deviations,

$$\frac{\delta S_1}{\delta l_a} = 0,\ \frac{\delta S_1}{\delta l_g} = 0,\ \frac{\delta S_1}{\delta l_k} = 0$$

and

$$\frac{\delta S_2}{\delta l_a} = 0,\ \frac{\delta S_2}{\delta l_g} = 0,\ \frac{\delta S_2}{\delta l_k} = 0$$

when $\dfrac{\delta S_1}{\delta l_a} = 0,$

$$\sum_1^n 2\left[l_a 2r\cos(\theta - \alpha) + l_g .2r\cos(\alpha - \gamma) + l_k - r^2\right].2r\cos(\theta - \alpha) = 0$$

or

$$l_a \sum_1^n \left[2r\cos(\theta - \alpha)\right]^2 + l_g \sum_1^n \left[2r\cos(\alpha - \gamma)\right]\left[2r\cos(\theta - \alpha)\right]$$

$$+ l_k \sum_1^n \left[2r\cos(\theta - \alpha)\right] = \sum \left[2r\cos(\theta - \alpha)\right] r^2$$

Similarly for $\dfrac{\delta S_1}{\delta l_g} = 0$ and $\dfrac{\delta S_1}{\delta l_k} = 0,$

$$l_a \sum_1^n \left[2r\cos(\theta - \alpha)\right]\left[2r\cos(\alpha - \gamma)\right] + l_g \sum_1^n \left[2r\cos(\alpha - \gamma)\right]^2$$

$$+ l_k \sum_1^n \left[2r\cos(\alpha - \gamma)\right] = \sum_1^n \left[2r\cos(\alpha - \gamma)\right] r^2$$

$$l_a \sum_1^n \left[2r\cos(\theta - \alpha)\right] + l_g \sum_1^n \left[2r\cos(\alpha - \gamma)\right] + l_k = \sum_1^n r^2$$

Inserting n values of r, α, θ (given) and one value of γ (assumed), l_a, l_g and l_k can be calculated by using the Cramer's rule, etc.

Similarly, using the conditions $\dfrac{\delta S_2}{\delta l_a} = 0, \dfrac{\delta S_2}{\delta l_g} = 0$ and $\dfrac{\delta S_2}{\delta l_k} = 0,$ the values of m_a, m_g and m_k can be found.

The rest of the procedure is as given earlier for three values of α, θ and r.

Example 5.12 ***Design a four-link mechanism to coordinate three positions of the input link with three positions of the coupler point, the data for which is given below:***

$\theta_1 = 110°$	$r_1 = 80$ mm	$\alpha_1 = 65°$
$\theta_2 = 77°$	$r_2 = 90$ mm	$\alpha_2 = 56°$
$\theta_3 = 50°$	$r_3 = 96$ mm	$\alpha_3 = 48°$

Assume the values of γ, ψ and δ_1 as 20°, 10° and 150° respectively.

Solution: The procedure is as follows:

(a) Solve the simultaneous Eqs (5.13), (5.14) and (5.15) and obtain

$l_a = 8.14$ $l_g = 38.57$ $l_k = 1115.3$

(b) Write three equations from Eq. (5.12) and obtain

$m_a = 0.009\,58$ $m_g = 0.0173$ $m_k = -3.045$

(c) Find two values of λ using Eq. (5.10).

$\lambda_1 = 945$ $\lambda_2 = 2001$

(d) Using $\lambda = 945$, obtain a, g and k from relations given in Eq. (5.9). Also, find e from the relation of Eq. (5.8).

$a = 17.2$ $g = 55$ $e = 39.4$

(e) Using three relations for β [Eq. (5.16)], find β_1, β_2, β_3.

$\beta_1 = 107.7°$ $\beta_2 = 97.5°$ $\beta_3 = 87.7°$

Thus, all the parameters for the loop *OABEO* are known.

(f) Obtain δ_2 and δ_3 from relations of Eqs (5.21) and (5.22).

$\delta_2 = 139.8°$ $\delta_3 = 130.1°$

The same procedure is adopted for the loop *ODCEO*.

(g) Solving Eqs (5.19) and (5.20), the following values are obtained:

$l_f = 71.15$ $l_h = 62.16$ $l_p = 1688$

$m_f = 0.0262$ $m_h = 0.00342$ $m_p = -2.21$

(h) Write three equations similar to Eq. (5.9) in λ' and obtain

$\lambda'_1 = -9507$ $\lambda'_2 = 5197$

When $\lambda' = -9507$; $f = -320.3$; $h = 29.7$ and $c = 355.2$

When $\lambda' = 5197$; $f = 65.1$; $h = 79.9$ and $c = 28.5$

(i) When $\lambda = 2001$, another set of a, g and e and two more sets of f, h and c are obtained.

```
Enter values oftb1, tb2, tb3, fl, r2,
r3, all, al2, al3, gamm, si, dell
1 1 0   7 7   5 0   8 0   9 0   9 6
65  56  48  20  10  150

   q      a      e      h      c       f

 54.95  17.20  39.40  20.69  355.21  -320.28

 54.95  17.20  39.40  79.91   28.50    65.04

 73.25  27.31  33.68  23.31   96.77   -47.12

 73.25  27.31  33.68 103.50   20.07    71.12
```

Fig. 5.40

Figure 5.40 shows the input and the four sets of values obtained by using the program of Fig. 5.39.

Figure 5.41 shows the solution obtained from the second set.

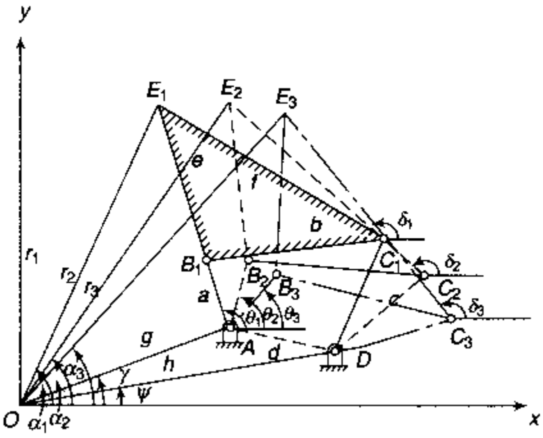


Fig. 5.41

5.11 MOTION GENERATION (RIGID-BODY GUIDANCE)

Assume that a rigid body *IJKL* is required to be guided through three finitely separated positions as shown in Fig. 5.42. The three positions of the body may be specified by taking any line on the body and marking a point E on the same. Then three positions of the point E may be specified by the radial distances from the assumed origin and its angular positions, i.e., by r_1, α_1; r_2, α_2, and r_3, α_3; and angular inclination of the line with the x-axis by the angles β_1, β_2 and β_3.

Now if it is assumed that the body is fixed to the coupler link, it becomes a problem similar to that of path generation except that now three values of β are known instead of θ. Thus, now angle θ can be eliminated from Eq. (5.5) and (5.6) instead of β. The equations formed are exactly the same if θ is replaced by β in Eqs (5.11) and (5.12). Also, as β is directly known, there is no need of using Eq. 5.16.

The program given in Fig. 5.43 solves this type of problem. The input variables are

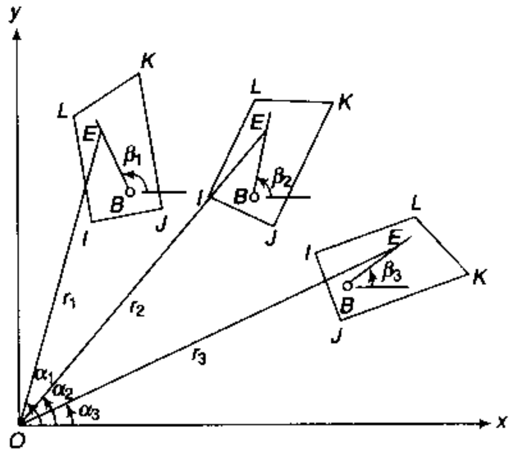


Fig. 5.4

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main ( )
 {
  FILE *fp;
  int k,j;
  float al,a2,a3,all,a22,a33,al2,a21,g12
  ,g21,e12,e21,akl,
  ak2,ak3,all,al2,al3,aa,gl,g2,g3,tll,t
  22,t33,tbl,tb2,
  tb3,gg,gamm,ss,si,dll,d22,d33,rl,r2,r
  3, pl,p2,p3,tl,t2,
  t3,cl,c2,c3,alg,ala,alk,ama,amg,amk,bb,cc,all,el,e2,
  squ,betl,bet2,bet3,pll,p22,p33,e3,gs,del,dell,del2,
  del3,rad;
  clrscr( ) ;
  printf("Enter values of tbl,tb2,tb3,rl,r2,r3,all,all,");
  printf("al2,al3,gamm,si,dell\n");
  scanf("%f%f%f%f%f%f%f%f%f%f%f%f%f",&tbl,&tb2,&tb3,&rl,&r2,
  &r3,&all,&al2,&al3,&gamm,&si,&dell);
  rad=4*atan(1)/180;
  tll=tbl*rad;
  t22=tb2*rad;
  t33=tb3*rad;
  all=all*rad;
  a22=al2*rad;
  a33=al3*rad;
  gg=gamm*rad;
  ss=si*rad;
  dll=dell*rad;
  for (j=O; j<3; j++)
  {
   pl=2*rl*cos(tll-all);
   p2=2*r2*cos(t22-a22);
   p3=2*r3*cos(t33-a33);
   tl=2*rl*cos(all-gg);
   t2=2*r2*cos(a22-gg);
   t3=2*r3*cos(a33-gg);
   cl=rl*rl;
   c2=r2*r2 ;
   c3=r3*r3;
    for(k=O;k<2;k++)
```

```
{
 del=p1*(t2-t3)+t1*(p3-p2)+(p2*t3-p3*t2);
 del1=c1*(t2-t3)+t1*(c3-c2)+(c2*t3-c3*t2);
 del2=p1*(c2-c3)+c1*(p3-p2)+(p2*c3-p3*c2);
 del3=p1*(t2*c3-t3*c2)+t1*(c2*p3-c3*p2)
 +c1*(p2*t3-p3*t2);
 ak1=del1/del;
 ak2=del2/del;
 ak3=del3/del;
  if (k==0)
  {
   ala=ak1;
   alg=ak2;
   alk=ak3;
   c1=2*cos(t11-gg);
   c2=2*cos(t22-gg):
   c3=2*cos(t33-gg):
  }
 }
ama=ak1;
amg=ak2:
amk=ak3:
aa=ama*amg:
bb=ala*amg+alg*ama-1:
cc=ala*alg;
squ=bb*bb-4*aa*cc:
 if (squ>0)
 {
  all=sqrt(squ):
  a11=(-bb-all)/(2*aa):
  a12=(-bb+all)/(2*aa):
  a1=ala+a11*ama:
  g1=alg+a11*amg:
  a2=ala+a12*ama;
  g2=alg+a12*amg:
  e1=sqrt(alk+a11*amk+a1*a1+g1*g1):
  e2=sqrt(alk+a12*amk+a2*a2+g2*g2):
  if(j==0) ( printf("       g       e      a")
         printf("   h         c     f\n"):   }
  if(j==1) (printf("%8.2f %8.2f %8.2f %8.2f %8.2f
  %8.2f \n",g12,a12,e12,g1,e1,a1):
  printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f \n",
  g12,a12,e12,g2,e2,a2): }
  if(j==2){printf("%8.2f %8.2f %8.2f %8.2f %8.2f
  %8.2f \n",g21,a21,e21,g1,e1,a1):
  printf("%8.2f %8.2f %8.2f %8.2f %8.2f %8.2f \n",
  g21,a21,e21,g2,e2,a2);}
  if (j==0)
  {
        g12=g1:
```

```
            a12=a1;
            e12=e1:
            g21=g2:
            a21=a2:
            e21=e2:
            gs=gg:
            p11=t11:
            p22=t22;
            p33=t33:
        }
        if(j==1)
        {
            g1=g21:
            a1=a21:
            e1=e21;
            t11=d11:
            t22=d22:
            t33=d33:
            gg=gs:
            t11=p11;
            t22=p22;
            t33=p33;
     }
    d22=d11+t22-t11;
    d33=d11 +t33-t11;
    a3=a2;
    g3=g2;
    e3=e2;
    p11=t11;
    p22=t22;
    p33=t33;
    t11=d11;
    t22=d22;
    t33=d33;
    gs=gg;
    gg=ss;
  }
          }
 getch( );
}
```

Fig. 5.43

The input variables are

`tb1, tb2, tb3`	angles β_1, β_2 and β_3 respectively (degrees)
`r1, r2, r3`	radial distances of point *E* from the origin (mm)
`a11, a12, a13`	angular position of point *E* (degrees)
`gamm`	assumed value of the angle γ (degrees)